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Three-dimensional stable matching with cyclic preferences

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Abstract

We consider stable three-dimensional matchings of three genders (3GSM). Alkan [Alkan, A., 1988. Non-existence of stable threesome matchings. Mathematical Social Sciences 16, 207–209] showed that not all instances of 3GSM allow stable matchings. Boros et al. [Boros, E., Gurvich, V., Jaslar, S., Krasner, D., 2004. Stable matchings in three-sided systems with cyclic preferences. Discrete Mathematics 286, 1–10] showed that if preferences are cyclic, and the number of agents is limited to three of each gender, then a stable matching always exists. Here we extend this result to four agents of each gender. We also show that a number of well-known sufficient conditions for stability do not apply to cyclic 3GSM. Based on computer search, we formulate a conjecture on stability of "strongest link" 3GSM, which would imply stability of cyclic 3GSM.

Keywords: Stable matching; 3GSM; Cyclic preferences; Balanced game; Effectivity function

JEL classification: C78

1. Introduction

The *stable marriage problem* is: Given a set of men and a set of women, find a matching that is stable in the sense that no man m and woman w who both prefer each other to their current partners in the matching. Gale and Shapley (1962) introduced this problem and gave a constructive proof of the existence of a stable matching for any combination of preferences. The theory of stable matchings has become an important subfield within game theory, as documented

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by the book of Roth and Sotomayor (1990), but the first book on the subject was written by famous computer scientist Donald E. Knuth (1976). Knuth lists a dozen suggested further directions for research, one of which is to investigate *three-dimensional stable matching*, say of women, men and dogs. Such a matching would be a partition of the agents into *triples* consisting of one agent of each type. A matching is *stable* if there is no *blocking triple*, i.e. a triple that all of its members would strictly prefer to the current matching. A matching is *strongly stable* if there is no *weakly blocking triple*, i.e. a triple strictly preferred by some member and weakly preferred by all members. We will follow Ng and Hirschberg (1991) and refer to a system of agents of three types, together with their preferences on triples, as an instance of *3GSM* (Three Gender Stable Marriage problem).

Alkan (1988), who seems to have been the first who published a result on 3GSM, found an instance where no stable matching exists. Ng and Hirschberg (1991) showed that a number of instances of 3GSM do not have any strongly stable matchings, and proved that the decision problem is NP-complete. As an open problem, Ng and Hirschberg mention the *cyclic* (or circular) 3GSM, where women care only about which man is in the triple, and similarly men care only about dogs, and dogs care only about women. The origin of this problem is attributed to Knuth. Recently, Boros et al. (2004) proved stability for cyclic 3GSM whenever $n \leq 3$, where n is the number of agents of each type. (More generally, their result says that for any integer $s \geq 2$, cyclic s-GSM has a stable matching whenever $n \leq s$.) They also show that their method of proof breaks down for larger values of n.

In this paper, we will extend the partial result of Boros et al. by proving stability of cyclic 3GSM for $n=4$. The proof, given in Section 3, is a quite technical case-by-case analysis that does not easily generalize to larger n. Therefore we also investigate whether stability of cyclic 3GSM would be implied by any of a number of well-known general sufficient conditions for stability (balancedness, effectivity function stability). In Section 4 we show that none of these conditions apply to cyclic 3GSM.

Another possible approach is to find a suitable relaxation of the cyclicity condition. Danilov (2003) proved stability of 3GSM under a certain acyclic lexicographic preference rule, where men base their preferences on triples in the first place on the woman in the triple (and in the second place on the dog), and women similarly are interested primarily in men. Boros et al. (2004) studied the lexicographic relaxation of cyclic preferences, where women care in the first place about men (and in the second place about dogs), and cyclically for men and for dogs. However, under this rule they found instances of 3GSM where no stable matching exists. In this paper we propose another relaxation of cyclicity: “strongest link 3GSM” (defined in the next section). Evidence from computer search leads us to conjecture that strongest link 3GSM always allows stable matchings, see Section 5.

2. Problem definition

We consider three sets of agents: W, M, D (for women, men and dogs). Let n be the maximal number of agents in a set; e.g. $n=3$ means that we have at most nine agents (three women, three men, three dogs). Without loss of generality we can assume that we have the maximal number n of agents of each gender, for otherwise we can just fill the ranks with dummy agents who everybody likes less than any real agent. A *triple* is an element of $W \times M \times D$, and a set of n disjoint triples is a *matching*.

Each woman w has a preference order, denoted by $>_w$, over the $2n$ agents in $M \cup D$. Analogously, men have preferences over $D \cup W$, and dogs over $W \cup M$. Preferences can equivalently be expressed as *ranks*, where the most preferred option is ranked one, etc. For each woman w we will derive a weak preference order $\succeq_w$ over $M \times D$, i.e. over all possible triples that include w. By "weak" we mean that a woman may be indifferent between some triples. $T \succ_w T'$ means that w strictly prefers T to T' whereas $T \succeq_w T'$ also includes the case of indifference. Analogously for men and dogs, let $\mathcal{P}$ be the set of all $(2n)!$ possible combinations of preference orders on agents. Let $\mathcal{PT}$ be the set of all $n^2!$ possible combinations of preference orders on triples. A *triple preference rule* is a mapping from $\mathcal{P}$ to $\mathcal{PT}$, i.e. a derivation of $\succ_a$ from $>_a$ for any agent a.

Given a triple preference rule, *stable matchings* and *blocking triples* are defined as in the Introduction.

The general problem is: Given a triple preference rule, does a stable matching exist for every instance of 3GSM of size n? We will mainly discuss the *cyclic* triple preference rule: Let $t = wmd$ and $T' = w'm'd'$ be two arbitrary triples. Then $T \succ_w T'$ if and only if $m >_w m'$, and cyclically for men and dogs.

3. Cyclic 3GSM is stable for $n \leq 4$

As promised in the Introduction, we shall prove the following result.

Theorem 1. *If $n \leq 4$, every instance of cyclic 3GSM has a stable matching.*

The number of possibilities is too large for an exhaustive search, see Section 5. Instead we have found a way to reduce the possibilities to a few cases.

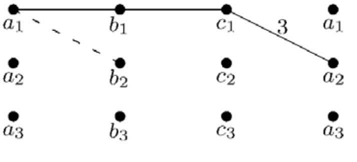
3.1. Notation

Given a matching, we say that an agent is *i-content* if she is matched (i.e. in the same triple) with the agent she ranks as number i.

We will denote triples by abc instead of wmd to signify that the first agent of the triple is not necessarily a woman, but that the order satisfies cyclicity (i.e. a's have preferences on b's, who have preferences on c's, who have preferences on a's).

A triple abc is called a $11i$-triple if a ranks b as number 1, b ranks c as number 1, and c ranks a as number i. We will use "she" as a generic pronoun.

We will often use a dot diagram to describe partial information about the preferences. Dots represent agents, dots in the same column belong to the same gender, and edges from one column to the next represent how the left agent ranks the right agent. A solid line means "rank 1", a dashed line means "rank 2" and an i-labeled line means "rank i". Here is an example:



The example diagram contains the information that a_1's second-best choice is b_2, and a_1's favorite is b_1 whose favorite is c_1 who ranks a_2 as number 3. To make the following pages more readable, we will omit the dot labels, always implicitly referring to the labeling above. (Of course, for $n = 4$ there will be an additional row $a_4b_4c_4a_4$ at the bottom.)

3.2. A lemma for n = 3

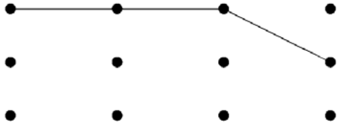
Boros et al. (2004) proved that there is always a stable matching for any instance of cyclic 3GSM with $n=3$. In order to get to the next value of n, we will first need a slight strengthening of their result.

Lemma 2. *For any cyclic 3GSM with $n=3$, and any agent x, there is a stable matching such that either x is 1-content, or x's favorite is 1-content and x is 2-content.*

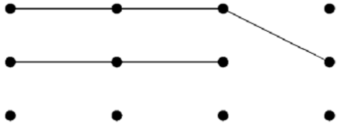
Proof. Without loss of generality we can assume that $x=a_1$, whose favorite is b_1, whose favorite is c_1. We will consider the two cases whether there is a 111-triple or not.

Suppose there is a 111-triple abc. Then pick that triple to the matching. Since a, b and c all are 1-content we can choose the other two triples however we like—the resulting matching will be stable anyway. If $b_1=b$ she is 1-content and we choose the remaining two triples so that a_1 gets her first or second choice. If $b_1 \neq b$ then $a_1 \neq a$ and we let the first triple be $a_1 b_1 c'$ where c' is b_1's first or second choice, and the second triple be the three remaining agents.

In the following we assume there is no 111-triple. Say, without loss of generality, that c_1's favorite is a_2.



By the assumption that there is no 111-triple, a_2's favorite is not b_1, so we can assume that it is b_2. Now, b_2's favorite is not c_1, so we can assume that it is c_2.



Now the matching $a_1b_1c_1$, $a_2b_2c_2$, $a_3b_3c_3$ is stable: a_1, a_2, b_1 and b_2 are 1-content, so a blocking triple must contain both a_3 and b_3 which are already together. □

3.3. Case-by-case analysis

We will divide all instances of cyclic 3GSM with $n=4$ into the following cases:

The 111-case: There is a 111-triple.
The 112-case: There is no 111-triple, but there is a 112-triple.
The 113-case: There is no 111- or 112-triple, but there is a 113-triple.
The 114-case: There is no 111-, 112- or 113-triple.

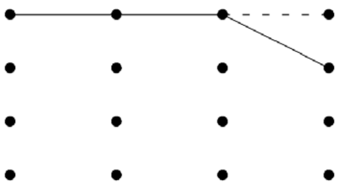
If there is no $11i$-triple for $i<j$ in a given matching, then we say that the matching satisfies the $11j$*-condition*.

The 111-case is trivial: remove the 111-triple and use Lemma 2 to find a stable matching of the remainder; then this matching together with the 111-triple constitutes a stable matching of the original instance, since the agents in the 111-triple are 1-content.

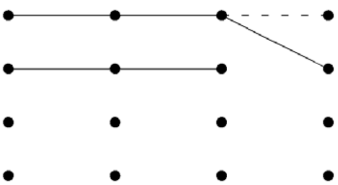
The 114-case is also simple: If there is an agent x who is the favorite of at least two agents, then x's favorite must rank some of these people as number 1, 2 or 3, and we get a 111-, 112-, or 113-triple. Thus, in the 114-case we know that no two agents have the same favorite. Simply let all a_i be 1-content in order to obtain a stable matching.

3.3.1. The 112-case

We have the following situation.



By the 112-condition, a_2's favorite is not b_1, so we can assume it is b_2. Now b_2's favorite is not c_1, so we can assume it is c_2.



We remove the triple $a_1b_1c_1$ for a while. By Lemma 2 there is a stable 3-matching of the remaining agents such that either a_2 is 1-content, or b_2 is 1-content and a_2 is 2-content. This 3-matching forms a 4-matching together with the triple $a_1b_1c_1$. We will show that this 4-matching is stable.

Suppose there is a blocking triple. It has to contain someone among a_1, b_1 and c_1 Since a_1 and b_1 are 1-content they do not belong to the blocking triple, so c_1 does. The only agent c_1 wants to switch to is a_2, so a_2 belongs to the blocking triple. Then a_2 cannot have her favorite b_2, so, by construction of the 3-matching, a_2 has her favorite among b_3 and b_4, and b_2 has c_2. So a_2 wants to switch only to b_2 or possibly b_1, both of which are 1-content.

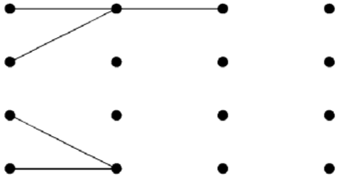
3.3.2. The 113-case

If some agent x is the favorite of at least three agents, then x's favorite must rank some of these agents as number 1 or 2, and we get a 111- or a 112-triple. Therefore, we can split the 113-case into two subcases:

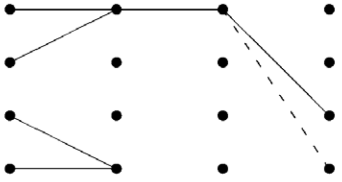
Subcase 1: Every agent is the favorite of either zero or two agents.

Subcase 2: There is an agent who is the favorite of exactly one agent, but every agent is the favorite of at most two agents.

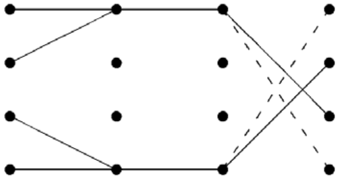
3.3.2.1. Subcase 1. Here we suppose that every agent is the favorite of either zero or two agents. Then we have the following situation.



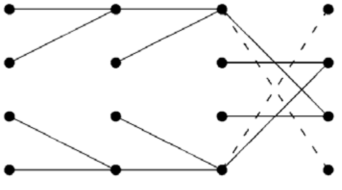
By the 113-condition we know that c_1 ranks a_1 and a_2 as numbers 3 and 4, so a_3 and a_4 must be numbers 1 and 2.



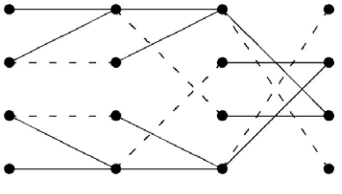
We see that b_4's favorite is not c_1, so we can assume it is c_4. Using the 113-condition again, we obtain the following.



Now we use that every favorite agent is the favorite of exactly two agents.



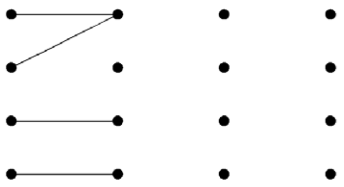
a_2's second-best choice cannot be b_3 or b_4, since that would violate the 113-condition, so it must be b_2. By applying the same reasoning to a_3, b_1 and b_4, we obtain the following diagram.



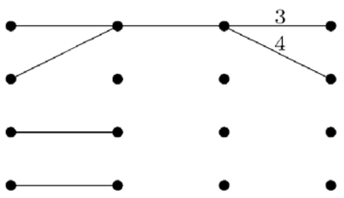
Now the matching $a_1b_1c_3$, $a_2b_2c_1$, $a_3b_3c_4$, $a_4b_4c_2$ is stable: a_1 and a_4 are 1-content, so a blocking triple must contain a_2 or a_3, say a_2. But a_2 wants to switch only to b_1 who wants to switch only to c_1 who already has got a_2. The same reasoning works for a_3.

3.3.2.2. Subcase 2. Here we suppose that there is an agent, say b_3, who is the favorite of exactly one agent, but every agent is the favorite of at most two agents.

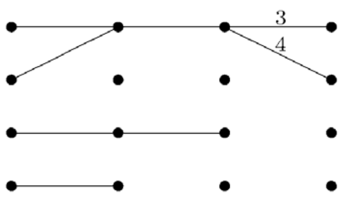
If every b_i is the favorite of exactly one agent, then it is trivial to find a stable matching: Just let all a_i be 1-content. So we assume there is a b_i, say b_1, who is the favorite of exactly two agents. Then we have the following situation:



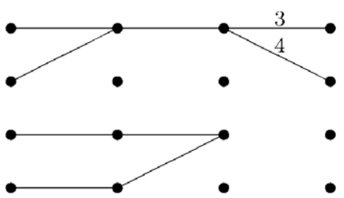
By the 113-condition, b_1's favorite must rank a_1 and a_2 as 3 and 4.



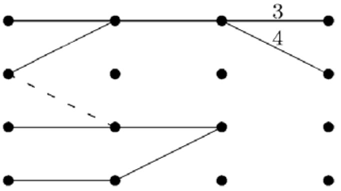
Now, b_3 or b_4 cannot have c_1 as a favorite, since c_1 ranks a_3 and a_4 as 1 and 2, and that would violate the 113-condition. Thus we can assume that b_3's favorite is, say, c_3.



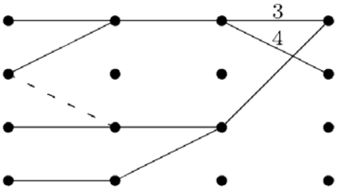
Suppose b_4 does not have c_3 as a favorite. Then we can assume that b_4's favorite is c_4 (remember that we know it is not c_1). But then $a_1b_1c_1$, $a_2b_2c_2$, $a_3b_3c_3$, $a_4b_4c_4$ is a stable matching since all a_i and b_i are 1-content, except a_2 and b_2 which are already together. Thus we can assume that b_4's favorite is c_3.



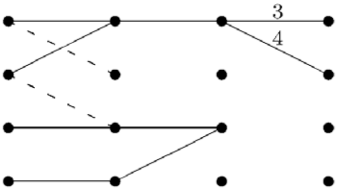
Again, form the matching $a_1b_1c_1$, $a_2b_2c_2$, $a_3b_3c_3$, $a_4b_4c_4$. What are the possible blocking triples? We observe that a_2 is the only a_i who is not 1-content, so a blocking triple must contain a_2. Since b_1 and b_3 are 1-content and a_2 already has got b_2, it follows that b_4 belongs to the blocking triple. We also see that c_1 cannot belong to the blocking triple, since c_1 already has a_1 whom she prefers to a_2. Thus, the only possible blocking triple is $a_2b_4c_3$. In that case, a_2 must prefer b_4 to b_2. In the same manner (using the matching $a_1b_1c_1$, $a_2b_2c_2$, $a_3b_3c_4$, $a_4b_4c_3$ instead) we deduce that a_2 prefers b_3 to b_2. This means that a_2's second-best choice is either b_3 or b_4. For symmetry reasons we can assume it is b_3.



By the 113-condition we know that c_3's favorite cannot be a_2, a_3 or a_4, so it must be a_1.



Using the 113-condition again, we see that a_1's second-best choice cannot be b_3 or b_4, so it must be b_2.



Now, the matching $a_1b_2c_2$, $a_2b_1c_1$, $a_3b_3c_3$, $a_4b_4c_4$ is stable: The only a_i who is not 1-content is a_1. She wants to switch only to b_1 who is 1-content.

4. Sufficient conditions for stability

The *core* of a game is the set of all outcomes for which no coalition of players can enforce another outcome that is preferable to all members of the coalition. For matching games it is easy to see that existence of a stable matching is equivalent to nonemptiness of the core (cf. Roth and Sotomayor, 1990). There exist several general approaches (i.e. sufficient conditions) to proving that a game has a nonempty core. The classic one is to show that the game is *balanced* in the sense of Scarf (1967). A reason to try this approach is that Quinzii (1984) showed (in a more general setting) that the usual two-dimensional matching game is balanced. However, we shall see below that our three-dimensional matching game is not balanced, making this approach hopeless.

There are also other types of sufficient conditions, framed in terms of "effectivity functions", cf. Boros and Gurvich (2000) for a good review of this theory. In essence, an effectivity function describes for any coalition which outcomes this coalition has the power to make sure that they would not occur. Peleg (1984) showed that "convex effectivity functions" are stable, i.e. the core is nonempty for any utility function. Boros and Gurvich (2000) describe a number of such conditions guaranteeing stability of effectivity functions. However, it is easy to see that no effectivity function argument can suffice to prove stability of cyclic 3GSM. The reason is that the effectivity function is the same as for unrestricted 3GSM, for which we already know (Alkan, 1988) that there are counterexamples to stability.

4.1. Cyclic 3GSM is not a balanced game

Following Scarf (1967) a collection C of triples is *balanced* if it is possible to find nonnegative real weights δ_T, for each triple T in C, such that, for each agent x,

$$\sum_{T:x\in T\in C} \delta_T = 1$$

A *utility vector* is a list where every agent has written down her utility goal, that is, how happy she hopes to become. A utility vector is *realizable* if there is a matching such that every agent reaches her utility goal. A utility vector is *realizable for a triple* if all agents in the triple would reach their utility goal if the triple were formed.

For the general definition of a *balanced game* we refer to Scarf (1967), who proved that the core of a balanced game is nonempty. Suffice it to say that cyclic 3GSM would be a balanced game if, for every balanced collection C of triples, a utility vector is realizable if it is realizable for every triple in C.

We will present a counterexample of size 3+3+3. Let C be the collection of triples corresponding to the shaded triangles in Fig. 1. This collection is balanced, since every agent belongs to exactly two triples (let all δ_T=1/2). Choose the preferences so that the edges in the figure correspond to rank 1 or 2. For example, a_1 will rank b_1 and b_3 as numbers 1 and 2 (in any order) and b_2 as number 3. Now consider the utility vector where every agent hopes to get at least her second-best choice. This is obviously realizable for every triple in C, so if the game were balanced, the utility vector would be realizable. Since every instance of "x ranks y as number 1 or 2" has a corresponding edge in the figure, a realization of the utility vector is equivalent to a disjoint family of triangles (not necessarily shaded) in the figure which covers all agents. But there is no such family: To cover a_2, either of the triangles $a_2b_1c_1$ and $a_2b_2c_2$ must belong to the

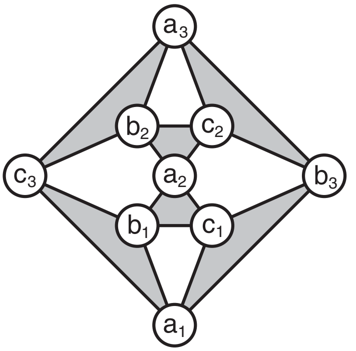


Fig. 1. An example showing that cyclic 3GSM is not a balanced game.

family. But none of the three triangles containing a_1 is disjoint with $a_2b_1c_1$, and none of the three triangles containing a_3 is disjoint with $a_2b_2c_2$.

5. Evidence from computer search

In this paper we have investigated Knuth's problem on stable three-dimensional matching under cyclic preferences, and improved on the partial solution from $n=3$ to 4 agents of each type. Presumably one reason that this problem is still open, after almost 30 years, is that it is difficult. We have seen that several general approaches to prove stability do not work.

For any given n, the number of instances is finite, and hence exhaustive search is theoretically possible. In practice, the number of possible combinations of preferences is simply too huge, $(2n)!^{3n}$, since each agent ranks all $2n$ agents of other genders. Even if isomorphic copies were deleted, the number of instances is large already for $n=3$ and daunting for $n>3$. Nonetheless, in order to heuristically investigate the stability of any given triple preference rule, we wrote a computer program that starts by generating a random instance of a given size $n \leq 5$. For this instance, each of the $n!^2$ possible matchings are checked for stability. The number of stable matchings is recorded. Then a local search for instances with fewer stable matchings is carried out as follows: Each of the $3n$ agents, in turn, changes its preference list to every possible alternative permutation. For each of these instances the number of stable matchings is computed, and search proceeds by steepest descent until a local minimum is found, after which the procedure is repeated from a new random starting point.

If an instance with zero stable matchings is found, we have a counterexample to the existence of stable matchings for the given rule. On the other hand, if no instance with zero stable matchings is found, then we have an indication that there is no counterexample. More specifically we obtain an indication of the minimum number of stable matchings.

In addition to the cyclic preference rule, we have studied the following two rules for triple preferences. Let $T=wmd$ and $T'=w'm'd'$ be two arbitrary triples.

- *Weakest link*: $T \succ_w T'$ if and only if $\min_w(m, d) >_w \min_w(m', d')$, and cyclically for men and dogs. Thus agents rank triples according to their least preferred partner, the weakest link.
- *Strongest link*: $T \succ_w T'$ if and only if $\max_w(m, d) >_w \max_w(m', d')$, and cyclically for men and dogs. Thus agents rank triples according to their most preferred partner, the strongest link.

Note that the cyclic rule can be derived as a special case both of the weakest link rule and the strongest link rule.

For $n=4$ the computer found counterexamples to stability under the weakest link rule, as well as for strong stability under the cyclic rule. All counterexamples are available from the authors.

However, quite extensive computer search for $n=4$ and $n=5$ supports the following conjecture.

Conjecture 3. *Every instance of strongest link 3GSM (with $n \geq 2$) has at least two stable matchings.*

This conjecture would imply stability of cyclic 3GSM. In fact, for cyclic 3GSM the computer always find many stable matchings, and it seems that the minimal number of stable matchings increases with n. We do not have enough evidence for a more precise conjecture.

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