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On the sign-imbalance of skew partition shapes

Jonas Sjöstrand

Department of Mathematics, Royal Institute of Technology, SE-100 44 Stockholm, Sweden

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Abstract

Let the *sign* of a skew standard Young tableau be the sign of the permutation you get by reading it row by row from left to right, like a book. We examine how the sign property is transferred by the skew Robinson–Schensted correspondence invented by Sagan and Stanley. The result is a remarkably simple generalization of the ordinary non-skew formula.

The sum of the signs of all standard tableaux on a given skew shape is the *sign-imbalance* of that shape. We generalize previous results on the sign-imbalance of ordinary partition shapes to skew ones.

1. Introduction

A *labelled poset* (P, ω) is an n-element poset P with a bijection $\omega : P \to [n] = \{1, 2, \ldots, n\}$ called the *labelling* of P. A *linear extension* of P is an order-preserving bijection $f : P \to [n]$. It is natural to define the *sign* of f as -1 to the power of the number of inversions with respect to the labelling, i.e., pairs $x, y \in P$ such that $\omega(x) < \omega(y)$ and $f(x) > f(y)$. The *sign-imbalance* $I_{P,\omega}$ of (P, ω) is the sum of the signs of all linear extensions of P. Note that $I_{P,\omega}$ is independent of the labelling ω up to sign. In this paper we will mainly discuss the square of sign-imbalances, and then we may drop the ω and write $I_P^2 = I_{P,\omega}^2$.

If $I_P^2 = 0$ the poset is *sign-balanced*. Such posets have been studied since 1989 by Ruskey [4, 5], Stanley [12], and White [13]. It is a vast subject however, and most of the work has been devoted to a certain class of posets: the partition shapes (or Young diagrams). Though no one so far has been able to completely characterize the sign-balanced partition shapes, this research direction has offered a lot of interesting results. Many people have studied the more general notion of sign-imbalance of partition shapes, among them Lam [2], Reifegerste [3], Sjöstrand [9], Shimozono and White [8], Stanley [12], and White [13].

E-mail address: jonass@kth.se.

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Young tableaux play a central role in the theory of symmetric functions (see [1]) and there are lots of useful tools for working with them that are not applicable to general posets. One outstanding tool is the Robinson–Schensted correspondence which has produced nice results also in the field of sign-imbalance; see [9,3,8].

As suggested in [9] a natural step from partition shapes towards more general posets would be to study *skew* partition shapes. They have the advantage of being surrounded by a well-known algebraic and combinatorial machinery just like the ordinary shapes, and possibly they might shed some light on the sign-imbalance of the latter ones as well. We will use a generalization of the Robinson–Schensted algorithm for skew tableaux by Sagan and Stanley [6].

In a recent paper [10, Theorem 4.3 and 5.7] Soprunova and Sottile show that $|I_{P,\omega}|$ is a lower bound for the number of real solutions to certain polynomial systems. Theorem 6.4 in [10] says that $|I_{P,\omega}|$ is the characteristic of the Wronski projection on certain projective varieties associated with P. When P is a skew partition shape this is applicable to skew Schubert varieties in Grassmannians (Richardson varieties).

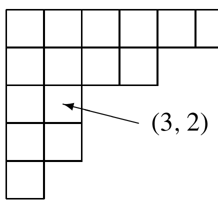
An outline of this paper:

- After some basic definitions in Section 2, in Section 3 we briefly recall Sagan and Stanley's skew RS-correspondence from [6].
- In Section 4 we state our main results without proofs and examine their connection to old results.
- In Sections 5 and 6 we prove our main theorems through a straightforward but technical analysis.
- In Section 7 we examine a couple of interesting corollaries to our main results. One corollary is a surprising formula for the square of the sign-imbalance of any ordinary shape.
- Finally, in Section 8 we suggest some future research directions.

2. Preliminaries

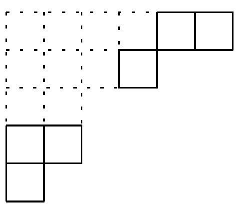
An (ordinary) *n-shape* $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a graphical representation (a Ferrers diagram) of an integer partition of $n = \sum_i \lambda_i$. We write $\lambda \vdash n$ or $|\lambda| = n$. The *coordinates* of a cell are the pair (r, c) where r and c are the row and column indices. Example:

$(6, 4, 2, 2, 1) =$



A shape μ is a *subshape* of a shape λ if $\mu_i \leq \lambda_i$ for all i. For any subshape $\mu \subseteq \lambda$ the *skew* shape λ/μ is λ with μ deleted. A *skew n-shape* λ/μ is a skew shape with n cells, and we write $\lambda/\mu \vdash n$ or $|\lambda/\mu| = n$. Here is an example of a skew 6-shape:

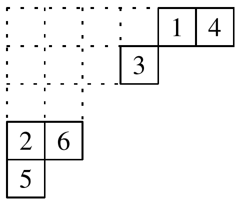
$(6, 4, 2, 2, 1)/(4, 3, 2) =$



A *domino* is a rectangle consisting of two cells. For an ordinary shape λ, let $v(\lambda)$ denote the maximal number of disjoint vertical dominoes that fit in the shape λ.

A *(partial) tableau* T on a skew n-shape λ/μ is a labelling of the cells of λ/μ with n distinct real numbers such that every number is greater than its neighbours above and to the left. We let $\sharp T = n$ denote the number of entries in T, and $\mathrm{PT}(\lambda/\mu)$ denote the set of partial tableaux on λ/μ.

A *standard tableau* on a skew n-shape is a tableau with the numbers $[n] = \{1, 2, \ldots, n\}$. We let $\mathrm{ST}(\lambda/\mu)$ denote the set of standard tableaux on the shape λ/μ. Here is an example:



The (skew) shape of a tableau T is denoted by $\operatorname{sh} T$. Note that it is not sufficient to look at the cells of T in order to determine its shape; we must think of the tableau as remembering its underlying skew shape. (For instance, $(6, 4, 2, 2, 1)/(4, 3, 2)$ and $(6, 4, 3, 2, 1)/(4, 3, 3)$ are distinct skew shapes that have the same set of cells.)

The *sign* of a number sequence $w_1 w_2 \cdots w_k$ is $(-1)^{\sharp\{(i,j): i<j,\, w_i > w_j\}}$, so it is $+1$ for an even number of inversions, -1 otherwise. The *inverse sign* is defined to be $(-1)^{\sharp\{(i,j): i<j,\, w_i < w_j\}}$.

The *sign* $\operatorname{sgn} T$ and the *inverse sign* $\operatorname{invsgn} T$ of a tableau T are the sign and the inverse sign, respectively, of the sequence you get by reading the entries row by row, from left to right and from top to bottom, like a book. Our example tableau has 4 inversions and 11 non-inversions, so $\operatorname{sgn} T = +1$ and $\operatorname{invsgn} T = -1$.

Definition 2.1. The sign-imbalance $I_{\lambda/\mu}$ of a skew shape λ/μ is the sum of the signs of all standard tableaux on that shape:

$$I_{\lambda/\mu} = \sum_{T \in \mathrm{ST}(\lambda/\mu)} \operatorname{sgn} T.$$

An empty tableau has positive sign and $I_{\lambda/\lambda} = I_\emptyset = 1$.

A *biword* π is a sequence of vertical pairs of positive integers $\pi = \begin{smallmatrix} i_1 i_2 \cdots i_k \\ j_1 j_2 \cdots j_k \end{smallmatrix}$ with $i_1 \le i_2 \le \cdots \le i_k$. We define the top and bottom lines of π by $\hat{\pi} = i_1 i_2 \cdots i_k$ and $\check{\pi} = j_1 j_2 \cdots j_k$. A *partial n-permutation* is a biword where in each line the elements are distinct and of size at most n. Let PS_n denote the set of partial n-permutations.

With each $\pi \in \mathrm{PS}_n$ we associate an ordinary n-permutation $\bar{\pi} \in S_n$ constructed as follows: First take the numbers among $1, 2, \ldots, n$ that do not belong to $\hat{\pi}$ and sort them in increasing order $a_1 < a_2 < \cdots < a_\ell$. Then sort the numbers among $1, 2, \ldots, n$ that do not belong to $\check{\pi}$ in increasing order $b_1 < b_2 < \cdots < b_\ell$. Now insert the vertical pairs $\begin{smallmatrix} a_r \\ b_r \end{smallmatrix}$, $1 \le r \le \ell$, into π so that the top line remains increasingly ordered (and hence must be $12 \cdots n$). The bottom line is a permutation (in single-row notation) which we denote as $\bar{\pi}$. Example: If $n = 5$ and $\pi = \begin{smallmatrix} 124 \\ 423 \end{smallmatrix}$ then $\bar{\pi} = 42135$.

In the following we let $\uplus$ denote disjoint union interpreted liberally. For instance, we will write $\check{\pi} \uplus T = [n]$ meaning that the set of numbers appearing in $\check{\pi}$ and the set of entries of the tableau T are disjoint and their union is $[n]$.

3. The skew RS-correspondence

In [6] Bruce Sagan and Richard Stanley introduced several analogues of the Robinson–Schensted algorithm for skew Young tableaux. Their main result is the following theorem.

Theorem 3.1 (*Sagan and Stanley; 1990*). *Let n be a fixed positive integer and α a fixed partition (not necessarily of n). Then there is bijection*

$$(\pi, T, U) \leftrightarrow (P, Q)$$

between $\pi \in \mathrm{PS}_n$ *with* $T, U \in \mathrm{PT}(\alpha/\mu)$ *such that* $\check{\pi} \uplus T = \hat{\pi} \uplus U = [n]$*, on the one hand, and* $P, Q \in \mathrm{ST}(\lambda/\alpha)$ *such that* $\lambda/\alpha \vdash n$*, on the other.*

Though we will assume detailed familiarity with it, we do not define the bijection here, but refer the reader to [6] for the original presentation.

4. Our results

In [9,3] the author and Astrid Reifegerste independently discovered the formula for sign transfer under the RS-correspondence:

Theorem 4.1 (*Reifegerste; Sjöstrand; 2003*). *Under the (ordinary) RS-correspondence* $\pi \leftrightarrow (P, Q)$ *we have*

$$\operatorname{sgn} \pi = (-1)^{v(\lambda)} \operatorname{sgn} P \operatorname{sgn} Q$$

where λ *is the shape of* P *and* Q.

Our main theorem is a generalization of this to Sagan and Stanley's skew RS-correspondence:

Theorem 4.2. *Under the skew RS-correspondence* $(\pi, T, U) \leftrightarrow (P, Q)$ *we have*

$$(-1)^{v(\lambda)} \operatorname{sgn} P \operatorname{sgn} Q = (-1)^{|\alpha|}(-1)^{v(\mu)+|\mu|} \operatorname{sgn} T \operatorname{sgn} U \operatorname{sgn} \bar{\pi}$$

where $\operatorname{sh} P = \operatorname{sh} Q = \lambda/\alpha$ *and* $\operatorname{sh} T = \operatorname{sh} U = \alpha/\mu$.

Note that if $\alpha = \emptyset$ the theorem reduces to Theorem 4.1.

Remark. If we specialize to the skew RS-correspondence $(\pi, T) \leftrightarrow P$ of involutions (see Corollary 3.4 in [6]), Theorem 4.2 gives that

$$(-1)^{v(\lambda)} = \operatorname{sgn} \bar{\pi} (-1)^{v(\mu)+|\mu|+|\alpha|},$$

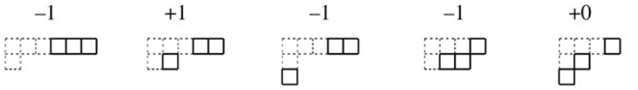
where $\operatorname{sh} P = \lambda/\alpha$ and $\operatorname{sh} T = \alpha/\mu$. This is also a simple consequence of Corollary 3.6 in [6] which is a generalization of a theorem by Schützenberger [7, page 127] (see also [11, exercise 7.28a]).

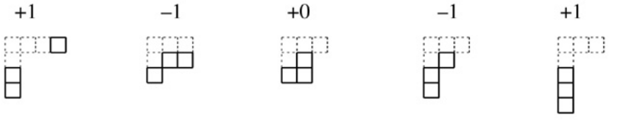
A fundamental application of Theorem 4.1 appearing in both [9] and [3] is the following theorem.

Theorem 4.3 (*Reifegerste; Sjöstrand; 2003*). *For all* $n \geq 2$

$$\sum_{\lambda \vdash n} (-1)^{v(\lambda)} I_\lambda^2 = 0.$$

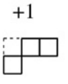
Let $\alpha = (3, 1)$ and $n = 3$. There are 10 skew shapes $\lambda/\alpha \vdash 3$. Here we have evaluated $(-1)^{\upsilon(\lambda)} I^2_{\lambda/\alpha}$ for each one of them:





(It happens that all these skew shapes have sign-imbalance 0 or 1, but in larger examples we would find much more exotic integers, like −7 for instance.) Now we compute $(-1)^{\upsilon(\mu)} I^2_{\alpha/\mu}$ for the two skew shapes $\alpha/\mu \vdash 2$:

Finally, there is only one skew shape $\alpha/\mu \vdash 3$:



We check that

$$\sum_{\lambda/\alpha \vdash 3} (-1)^{\upsilon(\lambda)} I^2_{\lambda/\alpha} = -2 = -1-1 = \sum_{\alpha/\mu \vdash 2} (-1)^{\upsilon(\mu)} I^2_{\alpha/\mu} - \sum_{\alpha/\mu \vdash 3} (-1)^{\upsilon(\mu)} I^2_{\alpha/\mu}.$$

Fig. 1. Example of Theorem 4.4.

We give a natural generalization of this using Theorem 4.2. It may be called a "sign-imbalance analogue" to Corollary 2.2 in [6].

Theorem 4.4. *Let α be a fixed partition and let n be a positive integer. Then*

$$\sum_{\lambda/\alpha \vdash n} (-1)^{v(\lambda)} I^2_{\lambda/\alpha} = \sum_{\alpha/\mu \vdash n} (-1)^{v(\mu)} I^2_{\alpha/\mu}$$

if n is even, and

$$\sum_{\lambda/\alpha \vdash n} (-1)^{v(\lambda)} I^2_{\lambda/\alpha} = \sum_{\alpha/\mu \vdash n-1} (-1)^{v(\mu)} I^2_{\alpha/\mu} - \sum_{\alpha/\mu \vdash n} (-1)^{v(\mu)} I^2_{\alpha/\mu}$$

if n is odd.

Fig. 1 gives an example. Observe that if $\alpha = \emptyset$ and $n \geq 2$ the theorem reduces to Theorem 4.3.

5. The proof of the main theorem

For a skew shape λ/μ, let

$$\operatorname{rsgn} \lambda/\mu := (-1)^{\sum_{(r,c)\in\lambda/\mu} (r-1)}.$$

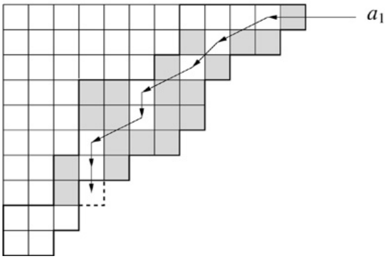


Fig. 2. External insertion of a_1. The shaded cells are counted by the sum $\sum_{i=2}^{r}(\beta_{i-1} - c_{i-1} + c_i - 1 - \gamma_i)$ in the proof.

For convenience, let $\operatorname{rsgn} T := \operatorname{rsgn} \operatorname{sh} T$ for a skew tableau T. Observe that for an ordinary shape λ we have $\operatorname{rsgn}\lambda = (-1)^{v(\lambda)}$.

For the sake of bookkeeping we will make two minor adjustments to the skew insertion algorithm that do not affect the resulting tableaux:

- Instead of starting with an empty Q-tableau, we start with the tableau U after multiplying all entries by ε. Here ε is a very small positive number.
- During an internal insertion a new cell with an integer b is added to the Q-tableau according to the usual rules. New additional rule: At the same time we remove the entry $b\varepsilon$ from the Q-tableau.

Consider the (adjusted) skew insertion algorithm starting with P-tableau $P_0 = T$ and Q-tableau $Q_0 = U\varepsilon$. After ℓ insertions (external or internal) we have obtained the tableaux P_ℓ and Q_ℓ. The following two lemmas state what happens when we make the next insertion.

Lemma 5.1. *Let $(P_{\ell+1}, Q_{\ell+1})$ be the resulting tableaux after external insertion of the number a_1 into (P_ℓ, Q_ℓ). Then*

$$\frac{\operatorname{sgn} P_{\ell+1}}{\operatorname{sgn} P_\ell} = \frac{\operatorname{sgn} Q_{\ell+1}}{\operatorname{sgn} Q_\ell}\frac{\operatorname{rsgn} Q_{\ell+1}}{\operatorname{rsgn} Q_\ell}(-1)^{\sharp Q_\ell}(-1)^m,$$

where m is the number of entries in P_ℓ that are less than a_1.

Proof. We insert the number a_1 which pops a number a_2 at $(1, c_1)$ which pops a number a_3 at $(2, c_2)$ and so on. Finally the number a_r fills a new cell (r, c_r); see Fig. 2.

For $2 \le i \le r$, the relocation of a_i multiplies the sign of the P-tableau by $(-1)^{\beta_{i-1}-c_{i-1}+c_i-1-\gamma_i}$, where $\operatorname{sh} P_\ell = \operatorname{sh} Q_\ell = \beta/\gamma$. Summation yields

$$\sum_{i=2}^{r}(\beta_{i-1} - c_{i-1} + c_i - 1 - \gamma_i) = -(c_1 - \gamma_1 + r - 2) + \sum_{i=1}^{r}(\beta_i - \gamma_i)$$

since $\beta_r = c_r - 1$. The placing of a_1 in the first row multiplies the sign of the P-tableau by $(-1)^{m-(c_1-1-\gamma_1)}$ where m is the number of entries in P_ℓ that are less than a_1. We get

$$\frac{\operatorname{sgn} P_{\ell+1}}{\operatorname{sgn} P_\ell} = (-1)^{m+1-r+\sum_{i=1}^{r}(\beta_i-\gamma_i)}.$$

Obviously

$$\frac{\operatorname{invsgn} Q_{\ell+1}}{\operatorname{invsgn} Q_\ell} = (-1)^{\sum_{i=1}^{r}(\beta_i-\gamma_i)}$$

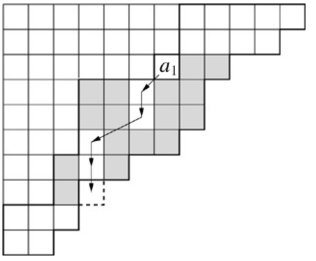


Fig. 3. Internal insertion starting with a_1. The shaded cells are counted by $\sum_{i=1}^{k}(\beta_{r+i-1} - c_{i-1} + c_i - 1 - \gamma_{r+i})$ in the proof.

and

$$\frac{\text{rsgn}\, Q_{\ell+1}}{\text{rsgn}\, Q_\ell} = (-1)^{r-1}.$$

Since $\text{sgn}\, R \,\text{invsgn}\, R = (-1)^{\binom{\sharp R}{2}}$ for any tableau R, we have

$$\frac{\text{invsgn}\, Q_{\ell+1}}{\text{invsgn}\, Q_\ell} = \frac{\text{sgn}\, Q_{\ell+1}}{\text{sgn}\, Q_\ell}(-1)^{\sharp Q_\ell}.$$

Combining the equations above proves the lemma. □

Lemma 5.2. *Let $(P_{\ell+1}, Q_{\ell+1})$ be the resulting tableaux after internal insertion of the entry a_1 at (r, c_0) into (P_ℓ, Q_ℓ). Then*

$$\frac{\text{sgn}\, P_{\ell+1}}{\text{sgn}\, P_\ell} = \frac{\text{sgn}\, Q_{\ell+1}}{\text{sgn}\, Q_\ell}\frac{\text{rsgn}\, Q_{\ell+1}}{\text{rsgn}\, Q_\ell}(-1)^{\sharp Q_\ell}.$$

Proof. During an internal insertion the entry a_1 at (r, c_0) pops a number a_2 at $(r + 1, c_1)$ which pops a number a_3 at $(r + 2, c_2)$ and so on. Finally the number a_k fills a new cell $(r + k, c_k)$; see Fig. 3.

For $1 \le i \le k$, the relocation of a_i multiplies the sign of the P-tableau by $(-1)^{\beta_{r+i-1} - c_{i-1} + c_i - 1 - \gamma_{r+i}}$, where $\text{sh}\, P_\ell = \text{sh}\, Q_\ell = \beta/\gamma$. Summation yields

$$\sum_{i=1}^{k}(\beta_{r+i-1} - c_{i-1} + c_i - 1 - \gamma_{r+i}) = -k + \sum_{j=r}^{r+k}(\beta_j - \gamma_j)$$

since $\beta_{r+k} = c_k - 1$ and $\gamma_r = c_0 - 1$.

What happens to the Q-tableau? According to our adjustments of the algorithm the entry $b\varepsilon$ at (r, c_0) is removed and the entry b is added at the new cell at $(r + k, c_k)$. Observe that $b\varepsilon$ is the smallest element in Q_ℓ; this is the very reason why we are making an internal insertion from its cell (r, c_0). Also note that b is the largest entry in $Q_{\ell+1}$. The transformation from Q_ℓ to $Q_{\ell+1}$ can be thought of as consisting of two steps: First we replace the entry $b\varepsilon$ by b, thereby changing the sign of the tableau by a factor $(-1)^{\sharp Q_\ell - 1}$. Then we move the b to the new cell at $(r + k, c_k)$, thereby changing the sign of the tableau by a factor

$$(-1)^{-1 + \sum_{j=r}^{r+k}(\beta_j - \gamma_j)}.$$

Now, after observing that

$$\frac{\operatorname{rsgn} Q_{\ell+1}}{\operatorname{rsgn} Q_\ell} = \frac{(-1)^{r+k}}{(-1)^r} = (-1)^k,$$

the lemma follows. □

Now we are ready to prove our main theorem.

Proof of Theorem 4.2. From Lemmas 5.1 and 5.2 we deduce by induction that

$$\frac{\operatorname{sgn} P}{\operatorname{sgn} T} = \frac{\operatorname{sgn} Q}{\operatorname{sgn} U}\frac{\operatorname{rsgn} Q}{\operatorname{rsgn} U}(-1)^{\sum_{\ell=0}^{n-1} \sharp Q_\ell}(-1)^{\sum m} \tag{1}$$

where $n = \sharp P$ and the last sum $\sum m$ is taken over all external insertions.

Let $t_1 < t_2 < \cdots < t_g$ and $u_1 < u_2 < \cdots < u_g$ be the entries of T and U, and write $\pi = \begin{smallmatrix} i_1 i_2 \cdots i_h \\ j_1 j_2 \cdots j_h \end{smallmatrix}$. Let π' be the permutation you get (in single-row notation) by preceding $\check{\pi}$ with the elements of T decreasingly ordered, i.e., $\pi' = t_g t_{g-1} \cdots t_1 j_1 j_2 \cdots j_h$. It is easy to see that the sum $\sum m$ equals the number of non-inversions of π', i.e. pairs $i < j$ such that $\pi'(i) < \pi'(j)$. This means that $(-1)^{\sum m} = \operatorname{invsgn} \pi'$.

What is the relationship between $\operatorname{invsgn} \pi'$ and $\operatorname{sgn} \bar{\pi}$?

Let us go from π' to $\bar{\pi}$ by a sequence of moves. Start with

$$\pi' = t_g t_{g-1} \cdots t_1 j_1 j_2 \cdots j_h.$$

Move the first entry t_g to position u_g:

$$\underbrace{t_{g-1} t_{g-2} \cdots t_1\ j_* \cdots j_*\ t_g}_{u_g \text{ entries}}\ j_* \cdots j_*.$$

(Here the symbolic indices $*$ should be replaced by the sequence $1, 2, \ldots, h$.) Next, move the entry t_{g-1} to position u_{g-1}:

$$\underbrace{t_{g-2} t_{g-3} \cdots t_1\ j_* \cdots j_*\ t_{g-1}}_{u_{g-1} \text{ entries}}\ j_* \cdots j_*\ t_g\ j_* \cdots j_*.$$

Continue until all elements of T are moved. The resulting permutation is $\bar{\pi}$. After analysing what the moves do to the sign of the permutation, we obtain

$$\operatorname{sgn} \bar{\pi} = (-1)^K \operatorname{sgn} \pi'$$

where

$$K = \sum_{i=1}^{g} (u_i - 1).$$

Note also that

$$\operatorname{invsgn} \pi' = (-1)^{\binom{n}{2}} \operatorname{sgn} \pi'.$$

Now look at

$$\sum_{\ell=0}^{n-1} \sharp Q_\ell.$$

If we define $K_\ell := \sharp\{b \in U : \ell < b\}$ we can write $\sharp Q_\ell = \ell + K_\ell$. Summation yields

$$\sum_{\ell=0}^{n-1} \sharp Q_\ell = \sum_{\ell=0}^{n-1} (\ell + K_\ell) = \binom{n}{2} + K + \sharp U.$$

Now we are ready to update (1):

$$\frac{\operatorname{sgn} P}{\operatorname{sgn} T} = \frac{\operatorname{sgn} Q}{\operatorname{sgn} U} \frac{\operatorname{rsgn} Q}{\operatorname{rsgn} U} \operatorname{sgn} \bar{\pi} (-1)^{\sharp U}.$$

There remains only some cleaning up. Observe that

$$\frac{\operatorname{rsgn} Q}{\operatorname{rsgn} U} = \frac{\operatorname{rsgn} \lambda/\alpha}{\operatorname{rsgn} \alpha/\mu} = \operatorname{rsgn} \lambda \operatorname{rsgn} \mu = (-1)^{v(\lambda)}(-1)^{v(\mu)}$$

and $\sharp U = |\alpha| - |\mu|$. This yields the result

$$(-1)^{v(\lambda)} \operatorname{sgn} P \operatorname{sgn} Q = (-1)^{|\alpha|}(-1)^{v(\mu)+|\mu|} \operatorname{sgn} T \operatorname{sgn} U \operatorname{sgn} \bar{\pi}. \quad \square$$

6. The proof of Theorem 4.4

In Theorem 3.1 we have adopted the original notation from Sagan and Stanley [6]. However, for some applications (and among them the forthcoming proof of Theorem 4.4) it is inconvenient to work with *partial* tableaux. For that matter we now present a simple bijection that will allow us to work with *standard* tableaux only.

Lemma 6.1. *Let n be a fixed positive integer and α and μ fixed partitions. Then there is a bijection $(\pi, T, U) \leftrightarrow (\tilde{\pi}, \tilde{I}, \tilde{T}, \tilde{U})$ between*

- *triples (π, T, U) such that $\pi \in \mathrm{PS}_n$, $T, U \in \mathrm{PT}(\alpha/\mu)$ and $\check{\pi} \uplus T = \hat{\pi} \uplus U = [n]$, and*
- *quadruples $(\tilde{\pi}, \tilde{I}, \tilde{T}, \tilde{U})$ such that $\tilde{\pi} \in S_n$, $\tilde{T}, \tilde{U} \in \mathrm{ST}(\alpha/\mu)$ and $\tilde{I} \subseteq [n]$ is the index set of an increasing subsequence of $\tilde{\pi}$ of length $|\alpha/\mu|$.*

This bijection has the following properties:

$$\begin{aligned} \tilde{\pi} &= \bar{\pi}, \\ \operatorname{sgn} \tilde{T} &= \operatorname{sgn} T, \\ \operatorname{sgn} \tilde{U} &= \operatorname{sgn} U. \end{aligned}$$

Proof. Given a quadruple $(\tilde{\pi}, \tilde{I}, \tilde{T}, \tilde{U})$, let the triple (π, T, U) be given by the following procedure: Write $\tilde{\pi}$ in biword notation and remove the vertical pairs corresponding to the increasing subsequence $\tilde{I}$. The resulting partial permutation is π. Order the elements in $\tilde{I}$ increasingly: $i_1 < i_2 < \cdots < i_k$. Now, for $1 \le j \le k$, replace the entry j in $\tilde{U}$ by i_j and replace the entry j in $\tilde{T}$ by $\tilde{\pi}(i_j)$. This results in U and T respectively. It is easy to see that this is indeed a bijection with the claimed properties. $\square$

Now we are ready to prove Theorem 4.4.

Proof of Theorem 4.4. Sum the equation of Theorem 4.2 over the whole domain of the skew RS-correspondence according to Theorem 3.1 in view of Lemma 6.1:

$$\sum_{\lambda/\alpha \vdash n} \sum_{P,Q \in \mathrm{ST}(\lambda/\alpha)} (-1)^{v(\lambda)} \operatorname{sgn} P \operatorname{sgn} Q$$
$$= \sum_{k=0}^{n} \sum_{\alpha/\mu \vdash k} \sum_{T,U \in \mathrm{ST}(\alpha/\mu)} \sum_{1 \le i_1 < \cdots < i_k \le n}$$
$$\sum_{\substack{\pi \in S_n \\ \pi(i_1) < \cdots < \pi(i_k)}} (-1)^{|\alpha|+v(\mu)+|\mu|} \operatorname{sgn} T \operatorname{sgn} U \operatorname{sgn} \pi.$$

Let LHS and RHS denote the left-hand side and the right-hand side of the equation above. The left-hand side trivially equals

$$\mathrm{LHS} = \sum_{\lambda/\alpha \vdash n} (-1)^{v(\lambda)} I^2_{\lambda/\alpha}.$$

The right-hand side is trickier. Fix $1 \le i_1 < i_2 < \cdots < i_k \le n$ and consider the sum

$$S := \sum_{\substack{\pi \in S_n \\ \pi(i_1) < \cdots < \pi(i_k)}} \operatorname{sgn} \pi.$$

- If $k = n$ clearly $S = 1$.
- If $k \le n - 2$ there are at least two integers $1 \le a < b \le n$ not contained in the sequence $i_1 < i_2 < \cdots < i_k$. The sign-reversing involution $\pi \mapsto \pi \cdot (a, b)$ (here (a, b) is the permutation that switches a and b) shows that $S = 0$.
- Suppose $k = n - 1$ and let a be the only integer in $[n]$ not contained in the sequence $i_1 < i_2 < \cdots < i_k$. We are free to choose $\pi(a)$ from $[n]$, but as soon as $\pi(a)$ is chosen, the rest of π must be the unique increasing sequence consisting of $[n] \setminus \pi(a)$ if π should contribute to S. The sign of π then becomes $(-1)^{\pi(a)-a}$ so

$$S = \sum_{i=1}^{n} (-1)^{i-a} = \begin{cases} 0 & \text{if } n \text{ is even,} \\ (-1)^{a-1} & \text{if } n \text{ is odd.} \end{cases}$$

In the case where n is odd and $k = n - 1$, the double sum

$$\sum_{1 \le i_1 < \cdots < i_k \le n} \sum_{\substack{\pi \in S_n \\ \pi(i_1) < \cdots < \pi(i_k)}} \operatorname{sgn} \pi = \sum_{a=1}^{n} (-1)^{a-1} = 1.$$

In summary we have shown

$$\sum_{1 \le i_1 < \cdots < i_k \le n} \sum_{\substack{\pi \in S_n \\ \pi(i_1) < \cdots < \pi(i_k)}} \operatorname{sgn} \pi = \begin{cases} 1 & \text{if } k = n, \\ 1 & \text{if } k = n - 1 \text{ and } n \text{ is odd,} \\ 0 & \text{if } k = n - 1 \text{ and } n \text{ is even,} \\ 0 & \text{if } k \le n - 2. \end{cases}$$

If n is even we finally obtain

$$\mathrm{RHS} = (-1)^{|\alpha|} \sum_{\alpha/\mu \vdash n} (-1)^{v(\mu)+|\mu|} \sum_{T,U \in \mathrm{ST}(\alpha/\mu)} \operatorname{sgn} T \operatorname{sgn} U = \sum_{\alpha/\mu \vdash n} (-1)^{v(\mu)} I^2_{\alpha/\mu}$$

since $(-1)^{|\alpha|+|\mu|} = (-1)^{|\alpha|-|\mu|} = (-1)^n = 1$.

Analogously, if n is odd we get

$$\begin{aligned}\text{RHS} &= (-1)^{|\alpha|} \sum_{\alpha/\mu \vdash n} (-1)^{v(\mu)+(|\alpha|-n)} \sum_{T,U \in \mathrm{ST}(\alpha/\mu)} \operatorname{sgn} T \operatorname{sgn} U \\ &\quad + (-1)^{|\alpha|} \sum_{\alpha/\mu \vdash n-1} (-1)^{v(\mu)+(|\alpha|-(n-1))} \sum_{T,U \in \mathrm{ST}(\alpha/\mu)} \operatorname{sgn} T \operatorname{sgn} U \\ &= \sum_{\alpha/\mu \vdash n-1} (-1)^{v(\mu)} I^2_{\alpha/\mu} - \sum_{\alpha/\mu \vdash n} (-1)^{v(\mu)} I^2_{\alpha/\mu}. \quad \square\end{aligned}$$

7. Specializations of Theorem 4.4

Apart from the special case $\alpha = \emptyset$, Theorem 4.4 offers a couple of other nice specializations if we choose the parameters α and n properly. First we obtain a surprising formula for the square of the sign-imbalance of any ordinary shape:

Corollary 7.1. *Let α be a fixed n-shape. Then*

$$I^2_\alpha = \sum_{\lambda/\alpha \vdash n} (-1)^{v(\lambda)} I^2_{\lambda/\alpha} = \sum_{\lambda/\alpha \vdash n+1} (-1)^{v(\lambda)} I^2_{\lambda/\alpha}$$

if n is even, and

$$I^2_\alpha = \sum_{\lambda/\alpha \vdash n-1} (-1)^{v(\lambda)} I^2_{\lambda/\alpha}$$

if n is odd.

Proof. First suppose n is even. Theorem 4.4 yields

$$\sum_{\lambda/\alpha \vdash n} (-1)^{v(\lambda)} I^2_{\lambda/\alpha} = \sum_{\alpha/\mu \vdash n} (-1)^{v(\mu)} I^2_{\alpha/\mu}.$$

The right-hand side consists of only one term, namely $(-1)^{v(\emptyset)} I^2_{\alpha/\emptyset} = I^2_\alpha$. From Theorem 4.4 we also get

$$\sum_{\lambda/\alpha \vdash n+1} (-1)^{v(\lambda)} I^2_{\lambda/\alpha} = \sum_{\alpha/\mu \vdash n} (-1)^{v(\mu)} I^2_{\alpha/\mu} - \sum_{\alpha/\mu \vdash n+1} (-1)^{v(\mu)} I^2_{\alpha/\mu}.$$

The second term of the right-hand side vanishes and the first term is I^2_α as before.

Now suppose n is odd. Then Theorem 4.4 yields

$$\sum_{\lambda/\alpha \vdash n-1} (-1)^{v(\lambda)} I^2_{\lambda/\alpha} = \sum_{\alpha/\mu \vdash n-1} (-1)^{v(\mu)} I^2_{\alpha/\mu}.$$

The right-hand side consists of only one term, namely $(-1)^{v((1))} I^2_{\alpha/(1)}$ which equals I^2_α since in an ordinary tableau the 1 is always located at (1, 1). $\square$

Next we present another generalization of Theorem 4.3.

Corollary 7.2. *Let α be a fixed n-shape. Then*

$$\sum_{\lambda/\alpha \vdash m} (-1)^{v(\lambda)} I^2_{\lambda/\alpha} = 0$$

for any integer $m \geq n+2$ if n is even, and for any integer $m \geq n$ if n is odd.

Proof. If m is even Theorem 4.4 yields

$$\sum_{\lambda/\alpha\vdash m}(-1)^{v(\lambda)}I^2_{\lambda/\alpha}=\sum_{\alpha/\mu\vdash m}(-1)^{v(\mu)}I^2_{\alpha/\mu}.$$

The right-hand side vanishes since $m > |\alpha|$.

If m is odd Theorem 4.4 yields

$$\sum_{\lambda/\alpha\vdash m}(-1)^{v(\lambda)}I^2_{\lambda/\alpha}=\sum_{\alpha/\mu\vdash m-1}(-1)^{v(\mu)}I^2_{\alpha/\mu}-\sum_{\alpha/\mu\vdash m}(-1)^{v(\mu)}I^2_{\alpha/\mu}.$$

If $m \geq n+2$ the right-hand side vanishes simply because $m-1 > |\alpha|$. Otherwise n is odd and the only remaining case is $m = n$. But then the right-hand side becomes $I^2_{\alpha/(1)} - I^2_\alpha = 0$. □

8. Future research

For an ordinary shape λ, let $h(\lambda)$ be the number of disjoint horizontal dominoes that fit in λ and let $d(\lambda)$ be the number of disjoint 2×2 squares (fourlings) that fit in λ.

In [9] the following theorem, conjectured by Stanley [12], was proved (part (a) was independently proved by Lam [2]):

Theorem 8.1 (*Stanley; Lam; Sjöstrand; 2003*).

(a) *For every* $n \geq 0$

$$\sum_{\lambda\vdash n} q^{v(\lambda)}t^{d(\lambda)}x^{h(\lambda)}I_\lambda=(q+x)^{\lfloor n/2\rfloor}.$$

(b) *If* $n \not\equiv 1 \pmod 4$

$$\sum_{\lambda\vdash n}(-1)^{v(\lambda)}t^{d(\lambda)}I^2_\lambda=0.$$

Part (b) is a strengthening of Theorem 4.3 and one might wonder whether there is a similar strengthening of Theorem 4.4 for skew shapes.

Part (a) is about signed sums of sign-imbalances without taking the square. From an RS-correspondence perspective it is unnatural not to take the square of the sign-imbalance since the P- and Q-tableaux come in pairs. In fact it might be argued that non-squared sign-imbalances are unnatural in all cases, because their sign is dependent on the actual labelling of the poset, i.e., it is important that we read the tableau as a book. However, part (a) in the theorem is still true (and there are even stronger theorems; see [9]) and it can be proved by means of the RS-correspondence as was done in [9]. This suggests that the skew RS-algorithm could be a useful tool for studying signed sums of non-squared sign-imbalances too.

As a tool for proving Theorem 8.1 the concept of *chess tableaux* was introduced in [9]. A chess tableau is a standard Young tableau where odd entries are located at an even Manhattan distance from the upper left cell of the shape, while even entries are located at odd distances. This notion of course generalizes to skew tableaux (in fact it generalizes to many other posets) and since it proved so useful in the study of sign-imbalance of ordinary shapes we think it will shed some light on the skew shapes as well.

Another direction of research is finding analogues to Theorem 4.2 for other variants of the RS-algorithm. For instance, in [6, Theorem 5.1] Sagan and Stanley present a generalization of their skew RS-correspondence where the condition that $\operatorname{sh} U = \operatorname{sh} T$ and $\operatorname{sh} P = \operatorname{sh} Q$ is relaxed. From that they are able to infer identities like

$$\sum_{\substack{\lambda/\beta \vdash n \\ \lambda/\alpha \vdash m}} f_{\lambda/\beta} f_{\lambda/\alpha} = \sum_{k \geq 0} \binom{n}{k} \binom{m}{k} k! \sum_{\substack{\alpha/\mu \vdash n-k \\ \beta/\mu \vdash m-k}} f_{\alpha/\mu} f_{\beta/\mu}$$

where $f_{\lambda/\mu} = \sharp \mathrm{ST}(\lambda/\mu)$. This correspondence may give interesting formulas for sums of products of sign-imbalances as well.

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References

[1] W. Fulton, Young Tableaux: With Applications to Representation Theory and Geometry, Cambridge University Press, Cambridge, 1997.
[2] T. Lam, Growth diagrams, Domino insertion and sign-imbalance, J. Combin. Theory Ser. A 107 (2004) 87–115. Also available at arXiv:math.CO/0308265.
[3] A. Reifegerste, Permutation sign under the Robinson–Schensted–Knuth correspondence, Ann. Comb. 8 (2004) 103–112. Also available at arXiv:math.CO/0309266.
[4] F. Ruskey, Generating linear extensions of posets by transpositions, J. Combin. Theory Ser. B 54 (1992) 77–101.
[5] F. Ruskey, Research problem 90, Discrete Math. 70 (1989) 111–112.
[6] B. Sagan, R. Stanley, Robinson–Schensted algorithms for skew tableaux, J. Combin. Theory Ser. A 55 (1990) 161–193.
[7] M.-P. Schützenberger, Quelques remarques sur une construction de Schensted, Math. Scand. 12 (1963) 117–128.
[8] M. Shimozono, D. White, A color-to-spin Domino Schensted algorithm, T. Electron. J. Combin. 8 (2001) R21.
[9] J. Sjöstrand, On the sign-imbalance of partition shapes, J. Combin Theory Ser. A 111 (2005) 190–203. Also available at arXiv:math.CO/0309231.
[10] E. Soprunova, F. Sottile, Lower bounds for real solutions to sparse polynomial systems, 2005. ArXiv:math.AG/0409504.
[11] R. Stanley, Enumerative Combinatorics, vol. 2, Cambridge University Press, New York, Cambridge, 1999.
[12] R. Stanley, Some remarks on sign-balanced and maj-balanced posets, 2003. ArXiv:math.CO/0211113.
[13] D. White, Sign-balanced posets, J. Combin. Theory Ser. A 95 (2001) 1–38.