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Bruhat intervals as rooks on skew Ferrers boards

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Abstract

We characterise the permutations π such that the elements in the closed lower Bruhat interval $[\mathrm{id}, \pi]$ of the symmetric group correspond to non-taking rook configurations on a skew Ferrers board. It turns out that these are exactly the permutations π such that $[\mathrm{id}, \pi]$ corresponds to a flag manifold defined by inclusions, studied by Gasharov and Reiner.

Our characterisation connects the Poincaré polynomials (rank-generating function) of Bruhat intervals with q-rook polynomials, and we are able to compute the Poincaré polynomial of some particularly interesting intervals in the finite Weyl groups A_n and B_n. The expressions involve q-Stirling numbers of the second kind, and for the group A_n putting $q = 1$ yields the poly-Bernoulli numbers defined by Kaneko.

Keywords: Coxeter group; Weyl group; Bruhat order; Poincaré polynomial; Rook polynomial; Partition variety

1. Introduction

Since its introduction in the 1930s the Bruhat order on Coxeter groups has attracted mathematicians from many areas. Geometrically it describes the containment ordering of Schubert varieties in flag manifolds and other homogeneous spaces. Algebraically it is intimately related to the representation theory of Lie groups. Combinatorially the Bruhat order is essentially the subword order on reduced words in the alphabet of generators of a Coxeter group.

The interval structure of the Bruhat order is geometrically very important and has been studied a lot in the literature. From a combinatorial point of view, as soon as there is a (graded) poset, the following three questions naturally arise about its intervals $[u, w]$ (and they will probably arise in the following order):

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(1) What is the rank-generating function (or Poincaré polynomial)

$$\mathrm{Poin}_{[u,w]}(q) = \sum_{v\in[u,w]} q^{\ell(v)}?$$

(2) What is the Möbius function $\mu(u, w)$?
(3) What can be said about the topology of the order complex of (u, w)?

The third question (and thus automatically the second question) was answered by Björner and Wachs [4] in 1982: The order complex of an open interval (u, w) is homeomorphic to the sphere $\mathbb{S}^{\ell(u,w)-2}$. The second question was answered already by Verma [23] in 1971: $\mu(u, w) = (-1)^{\ell(u,w)}$. However, the first question is still a very open problem!

For the whole poset Poin(q) was computed by Steinberg [22], Chevalley [6], and Solomon [21]. Really small intervals (of length $\leqslant 7$ in A_n and $\leqslant 5$ in B_n and D_n) were completely classified by Hultman [13] and Incitti [14]. Lower intervals of 312-avoiding permutations in A_n were classified by Develin [7] (though he did not compute their Poincaré polynomials), and for a general lower interval $[\mathrm{id}, w]$ in a crystallographic Coxeter group, Björner and Ekedahl [3] showed that the coefficients of $\mathrm{Poin}_{[\mathrm{id},w]}(q)$ are partly increasing. Reading [20] studied the cd-index and obtained a recurrence relation for the Poincaré polynomials of the intervals in any Coxeter group [19]; however he did not compute these for any particular intervals. Apart from this, virtually nothing seems to be known.

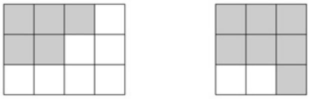
The aim of this paper is to start filling the hole and at least gain some understanding of the rank-generating function of a family of intervals in finite Weyl groups. To this end we present a connection between the Poincaré polynomial Poin(q) and rook polynomials, making it possible to compute Poin(q) for various interesting intervals. Our approach is partly a generalisation of the notion of *partition varieties* introduced by Ding [8] to what may be called *skew partition varieties*.

We characterise the permutations π such that the elements in the closed lower Bruhat interval $[\mathrm{id}, \pi]$ of the symmetric group correspond to non-taking rook configurations on a skew Ferrers board. It turns out that these are exactly the permutations π such that $[\mathrm{id}, \pi]$ corresponds to a flag manifold defined by inclusions, studied by Gasharov and Reiner [11]. We will elaborate over this interesting connection later on.

The paper is composed as follows. In Section 2 we give a short introduction to rook polynomials before presenting our results in Section 3. In Section 4 we present the connection between rook polynomials and Poincaré polynomials and prove our main theorem. In Sections 5 and 6 we apply our main theorem to intervals in the symmetric group A_n. As a by-product a Stirling number identity pops up at the end of Section 6. (This identity was found by Arakawa and Kaneko [1] in 1999 and its connection to Bruhat intervals was found by Launois [16] in 2005.) In Section 7 we apply our main theorem to the hyperoctahedral group B_n. Finally, in Section 8 we discuss further research directions and suggest some open problems.

2. Rook polynomials

Let A be a zero–one matrix and put rooks on some of the one-entries of A. If no two rooks are in the same row or column we have a (*non-taking*) *rook configuration on* A, and we say that A *covers* the rook configuration. In the literature, A is sometimes called a board and is often depicted by square diagrams like those in Fig. 1. For convenience we will simultaneously think



i	$r_i(\lambda)$	$c_i(\lambda)$	$r_i(\mu)$	$c_i(\mu)$
1	3	2	3	2
2	2	2	3	2
3	0	1	1	3
4		0		

Fig. 1. Square diagrams of the left-aligned Ferrers matrix $\lambda = \begin{pmatrix} 1\,1\,1\,0 \\ 1\,1\,0\,0 \\ 0\,0\,0\,0 \end{pmatrix}$ and the right-aligned Ferrers matrix $\mu = \begin{pmatrix} 1\,1\,1 \\ 1\,1\,1 \\ 0\,0\,1 \end{pmatrix}$. The row and column lengths are given by the table to the right.

of A as the set of its one-entries, and write for instance $(i,j) \in A$ if $A_{i,j} = 1$ and use notation like $A \cap B$. Sometimes we will refer to the entries of A as *cells*.

Let $A^{\updownarrow}$ and $A^{\circlearrowleft}$ denote reflecting the matrix upside down respectively rotating it 180 degrees, i.e. $A^{\updownarrow}_{i,j} = A_{m-i+1,j}$ and $A^{\circlearrowleft}_{i,j} = A_{m-i+1,n-j+1}$ if A is an $m \times n$ matrix. Define $\pi^{\updownarrow}$ and $\pi^{\circlearrowleft}$ similarly for rook configurations π.

The number of rook configurations on A with k rooks is called the *kth rook number of A* and is denoted by R^A_k. Given a non-negative integer n, following Goldman et al. [12] we define *the nth rook polynomial of A* as

$$\hat{R}^A_n(x) \stackrel{\text{def}}{=} \sum_{k=0}^{n} R^A_{n-k} x(x-1)\cdots(x-k+1).$$

Note that $\hat{R}^A_n(0) = R^A_n$.

A zero–one matrix λ is a *left-aligned* (*respectively right-aligned*) *Ferrers matrix* if every one-entry has one-entries directly to the left (respectively to the right) and above it (provided these entries exist). The number of ones in the ith row (respectively column) of λ is denoted by $r_i(\lambda)$ (respectively $c_i(\lambda)$). Figure 1 shows an example.

From [12] we have the following theorem.

Theorem 1. *(Goldman et al.) Let λ be a right-aligned Ferrers matrix of size $m \times n$. Then*

$$\hat{R}^\lambda_n(x) = \prod_{j=1}^{n} \bigl(x + c_j(\lambda) - j + 1\bigr).$$

Given a rook configuration $\mathcal{A}$ on A, define the statistics $\mathrm{inv}_A(\mathcal{A})$ to be the number of (not necessarily positive) cells of A with no rook weakly to the right in the same row or below in the same column. In the special case where A is an $n \times n$ matrix and $\mathcal{A}$ has n rooks, $\mathrm{inv}_A(\mathcal{A})$ becomes the number of inversions of the permutation π given by $\pi(i) = j \Leftrightarrow (i,j) \in \mathcal{A}$, where i is the row index and j is the column index.

Next, (almost) following Garsia and Remmel [10], we define the *kth q-rook number of A* as

$$R^A_k(q) = \sum_{\mathcal{A}} q^{\mathrm{inv}_A \mathcal{A}}$$

where the sum is over all rook configurations on A with k rooks. Given a non-negative integer n, the *nth q-rook polynomial of A* is defined as

$$\hat{R}^A_n(x;q) \stackrel{\text{def}}{=} \sum_{k=0}^{n} R^A_{n-k}(q)[x]_q[x-1]_q\cdots[x-k+1]_q.$$

Here $[x]_q \stackrel{\text{def}}{=} 1 + q + q^2 + \cdots + q^{x-1} = (1 - q^x)/(1 - q)$ is the q-analogue of x. Observe that putting $q = 1$ yields the ordinary rook numbers and polynomials.

Garsia and Remmel showed that Theorem 1 has a beautiful q-analogue:

Theorem 2. *(Garsia, Remmel) Let A be a left-aligned Ferrers matrix of size $m \times n$. Then*

$$\hat{R}_n^A(x; q) = q^z \prod_{j=1}^{n} \bigl[x + c_j(A) + j - n\bigr]_q$$

where z is the number of zero-entries in A.

Let $[n]!_q \stackrel{\text{def}}{=} [1]_q[2]_q \cdots [n]_q$.

Corollary 3. *For the $n \times n$ square matrix $J^{n,n}$ with ones everywhere, the nth q-rook number is*

$$R_n^{J^{n,n}}(q) = [n]!_q.$$

(This is the Poincaré polynomial of the whole Bruhat order on the symmetric group.)

Let T_n denote the $n \times n$ zero–one matrix with ones on and above the secondary diagonal, i.e. $(T_n)_{i,j} = 1 \Leftrightarrow i \leqslant n - j + 1$. In [10, p. 248] it is proved that

$$R_k^{T_n}(q) = q^{\binom{n}{2}} S_{n+1,n+1-k}(q) \tag{1}$$

where $S_{n,k}(q)$ is the *q-Stirling number* defined by the recurrence

$$S_{n+1,k}(q) = q^{k-1} S_{n,k-1}(q) + [k]_q S_{n,k}(q), \quad \text{for } 0 \leqslant k \leqslant n,$$

with the initial conditions $S_{0,0}(q) = 1$ and $S_{n,k}(q) = 0$ for $k < 0$ or $k > n$.

3. Results

A *skew Ferrers matrix* λ/μ is the difference $\lambda - \mu$ between a Ferrers matrix λ and an equally aligned componentwise smaller Ferrers matrix μ. If λ and μ are left-aligned, then λ/μ is also said to be left-aligned, and if λ and μ are right-aligned, so is λ/μ.

Let $\mathfrak{S}_n$ denote the symmetric group. We will often write permutations $\pi \in \mathfrak{S}_n$ in one-row notation: $\pi = \pi_1\pi_2 \cdots \pi_n$ where $\pi_i = \pi(i)$. For $\pi \in \mathfrak{S}_n$ and $\rho \in \mathfrak{S}_k$ we say that π *contains the pattern* ρ if there exist indices $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$ such that $\pi(i_r) < \pi(i_s)$ if and only if $\rho(r) < \rho(s)$. Otherwise π is said to *avoid* the pattern.

For any zero–one $n \times n$ matrix A, let $\mathfrak{S}(A)$ be the set of rook configurations on A with n rooks. We will identify such a rook configuration with a permutation $\pi \in \mathfrak{S}_n$ so that $\pi(i) = j$ if and only if there is a rook at the square (i, j), where i is the row index and j is the column index.

For a permutation $\pi \in \mathfrak{S}_n$, let the *right* (*respectively left*) *hull* $H_R(\pi)$ (*respectively* $H_L(\pi)$) *of* π be the smallest right-aligned (respectively left-aligned) skew Ferrers matrix that covers π. Figure 2 shows an example.

For the definition of Bruhat order and a general treatment of Coxeter groups from a combinatorialist's viewpoint, we refer to Björner and Brenti [2].

Our main result is the following theorem and its corollary.

Theorem 4. $\mathfrak{S}(H_R(\pi))$ *equals the lower Bruhat interval* $[\mathrm{id}, \pi]$ *in* $\mathfrak{S}_n$ *if and only if* π *avoids the patterns* 4231, 35142, 42513, *and* 351624.

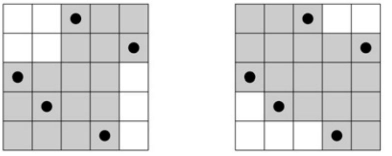


Fig. 2. The shaded regions show the left respectively right hull of the permutation 35124.

Upon discovering the above pattern avoidance condition, we searched the "Database of Permutation Pattern Avoidance" at http://www-math.mit.edu/~bridget/patterns.html for possible matches. It turned out that the permutations π in Theorem 4 are exactly the ones such that the Schubert variety corresponding to the interval $[\mathrm{id}, \pi]$ is *defined by inclusions* in the sense of Gasharov and Reiner [11] according to their Theorem 4.2. We will try to explain very briefly what this means.

A subset of the partial flag manifold (the set of flags of linear subspaces of $\mathbb{C}^n$)

$$\left\{0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_s \subsetneq \mathbb{C}^n\right\}$$

is called a Schubert variety if it is defined by inequalities of the form

$$\dim V_i \cap \mathbb{C}^e \geqslant r.$$

If we only allow $r = \dim V_i$ or $r = e$, the inequalities can be written as inclusions,

$$V_i \subseteq \mathbb{C}^e,$$
$$\mathbb{C}^{e'} \subseteq V_j,$$

and the Schubert variety is said to be *defined by inclusions*. Gasharov and Reiner give a simple presentation for the integral cohomology ring of smooth Schubert varieties, generalising Borel's presentation for the cohomology of the partial flag manifold itself. In their paper, it turns out that this presentation holds for a larger class of subvarieties of the partial flag manifolds, namely the ones defined by inclusion.

They also characterise the Schubert varieties defined by inclusions combinatorially by the same pattern avoidance condition as we have in Theorem 4. After learning this, we were able to write our proof of Theorem 4 (in Section 4) so that it converges with theirs at the end, and we will discuss later whether there is a more direct connection between our results.

Theorem 4 has the following useful corollary.

Corollary 5. *Let $u, w \in \mathfrak{S}_n$ and suppose w and $u^{\updownarrow}$ both avoid the patterns* 4231*,* 35142*,* 42513*, and* 351624*. Then the following holds.*

(1) *$\mathfrak{S}(H_R(w) \cap H_L(u))$ equals the Bruhat interval $[u, w]$.*
(2) *The Poincaré polynomial* $\mathrm{Poin}_{[u,w]}(q)$ *of $[u, w]$ equals the q-rook number $R_n^{H_R(w)\cap H_L(u)}(q)$.*
(3) *In particular, the number of elements in $[u, w]$ equals the ordinary rook number $R_n^{H_R(w)\cap H_L(u)}$.*

Proof. Once we observe that $H_L(u) = H_R(u^{\updownarrow})^{\updownarrow}$ and recall that flipping the rook configurations upside down is an antiautomorphism on the Bruhat order on $\mathfrak{S}_n$, the corollary follows directly from Theorem 4. □

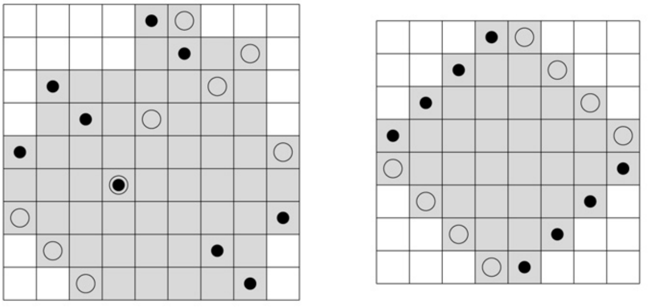


Fig. 3. Left: The permutations $u = 562314978$ (dots) and $w = 687594123$ (circles) in $\mathfrak{S}_9$ satisfy the pattern condition in Corollary 5, so the interval $[u, w]$ consists of precisely the permutations that fit inside the shaded region $H_R(w) \cap H_L(u)$. Right: $H_R(56781234) \cap H_L(43218765)$ is the Aztec diamond of order 4. (In fact it follows from part (3) of Corollary 5 that there are 2^n elements in the interval $[w^{\updownarrow}, w]$ in A_{2n-1}, where $w = \max A_{2n-1}^{S\setminus\{s_n\}}$.)

Remark 6. Since applying the upside down arrow $u^{\updownarrow}$ simply means reversing the entries of the permutation u in one-row notation, the condition on u in the theorem can be alternately stated by requiring u to avoid the patterns 1324, 24153, 31524, and 426153.

Remark 7. If π is 231-avoiding, then $[\mathrm{id}, \pi] = \mathfrak{S}(B)$ where B is the smallest right-aligned Ferrers matrix that covers π. In this case Ding [8] coined the name *partition variety* for the Schubert variety corresponding to the Bruhat interval $[\mathrm{id}, \pi]$ in $\mathfrak{S}_n$. Thus it would be logical to coin the name *skew partition variety* for a Schubert variety corresponding to an interval $[\mathrm{id}, \pi]$ such that $[\mathrm{id}, \pi] = \mathfrak{S}(H_R(\pi))$.

Figure 3 shows two examples of the corollary.

Non-trivial application of the above result yields the Poincaré polynomial of some particularly interesting intervals in finite Weyl groups.

For a Coxeter system (W, S) and a subset $J \subseteq S$ of the generators, let W_J denote the parabolic subgroup generated by J. Each left coset $wW_J \in W/W_J$ has a unique representative of minimal length, see [2, Corollary 2.4.5]. The system of such minimal coset representatives is denoted by W^J, and the Bruhat order on W restricts to an order on W^J.

We will deal with two infinite families of finite Coxeter systems, namely the symmetric groups A_n and the hyperoctahedral groups B_n. Their Coxeter graphs are depicted in Fig. 4.

For type A we have the following result.

Theorem 8. *Let w be the maximal element of $A_{n-1}^{S\setminus\{s_k\}}$. Then the Poincaré polynomial of the Bruhat interval* $[\mathrm{id}, w]$ *is*

$$\mathrm{Poin}_{[\mathrm{id},w]}(q) = q^{(n-k)k} \sum_{i=0}^{k} S_{k+1,i+1}(1/q) S_{n-k+1,i+1}(1/q) [i]!_q^2 q^i.$$

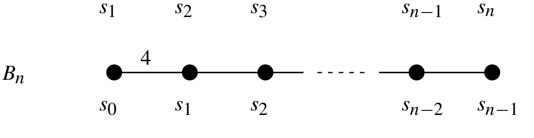


Fig. 4. The Coxeter graphs of A_n and B_n.

The special case where n is even, $k = n/2$, and $q = 1$ follows from Exercise 4.36 in Lovász [17] once one knows that the set of permutations he describes is a Bruhat interval (which is Exercise 2.6 in [2]). See also Theorem 3 in Vesztergombi [24].

For type B the corresponding result looks like this:

Theorem 9. *Let w be the maximal element of $B_n^{S\setminus\{s_0\}}$. Then the Poincaré polynomial of the Bruhat interval* [id, w] *is*

$$\mathrm{Poin}_{[\mathrm{id},w]}(q) = q^{\binom{n+1}{2}} \sum_{i=0}^{n} S_{n+1,i+1}(1/q)[i]!_q.$$

We also present a recurrence relation for computing the number of elements in the Bruhat interval [id, w] of A_{n-1} where w is any element in $A_{n-1}^{S\setminus\{s_k\}}$. As a by-product we obtain a new proof of the following Stirling number identity due to Arakawa and Kaneko [1].

Theorem 10. *Let w be the maximal element of $A_{n-1}^{S\setminus\{s_k\}}$. Then the number of elements in the Bruhat interval* [id, w] *is*

$$\mathrm{Poin}_{[\mathrm{id},w]}(1) = \sum_{i=0}^{k} S_{k+1,i+1} S_{n-k+1,i+1} i!^2 = (-1)^k \sum_{i=0}^{k} (-1)^i (i+1)^{n-k} i! S_{k,i} = B_n^{k-n}$$

where $S_{n,k}$ are the Stirling numbers of the second kind, and B_n^k are the poly-Bernoulli numbers *defined by Kaneko* [15].

Remark 11. Arakawa and Kaneko did not make the connection to Bruhat intervals but that part of Theorem 10 was proved by Launois [16] in 2005.

From Kaneko's work [15, p. 223] we can compute the exponential bivariate generating function for $\mathrm{Poin}_{[\mathrm{id},w]}(1)$ where w is the maximal element of $A_{n-1}^{S\setminus\{s_k\}}$,

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \mathrm{Poin}_{[\mathrm{id},w]}(1) \frac{x^n}{n!} \frac{y^{n-k}}{(n-k)!} = \frac{e^{x+y}}{e^x + e^y - e^{x+y}}.$$

The poly-Bernoulli numbers have the sequence number A099594 in Sloane's On-Line Encyclopedia of Integer Sequences.

4. Skew Ferrers matrices and Poincaré polynomials

In this section we make a connection between Poincaré polynomials and rook polynomials, and prove Theorem 4. To understand why one might even contemplate a theorem like this, recall that the *rank* or the *length* of a permutation is given by its inversion number, so q-rook numbers and Poincaré polynomials in $\mathfrak{S}_n$ count by the same statistics.

Proposition 12. *If λ/μ is a right-aligned skew Ferrers matrix of size $n \times n$, then $\mathfrak{S}(\lambda/\mu)$ is an order ideal in the Bruhat order of $\mathfrak{S}_n$.*

Proof. The Bruhat order is the transitive closure of the directed Bruhat graph whose edges correspond to length-increasing transpositions (see e.g. [2, Section 2.1]). Thus it suffices to show

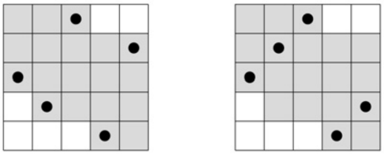


Fig. 5. The permutation 35124 to the left becomes 32154 to the right after exchanging rows 2 and 4. We do not leave the shaded region λ/μ by this operation.

that we cannot leave λ/μ by a transposition going down in the Bruhat order. In other words, if π is a rook configuration on λ/μ with n rooks, and $\pi_i > \pi_{i'}$ with $i < i'$, then exchanging rows i and i' yields a rook configuration which is covered by λ/μ. This is obviously true, as we can see in Fig. 5. □

For $\pi \in \mathfrak{S}_n$ and $i, j \in [n] = \{1, 2, \ldots, n\}$, let

$$\pi[i, j] \stackrel{\text{def}}{=} \left|\left\{a \in [i]\colon \pi(a) \geqslant j\right\}\right|.$$

In other words $\pi[i, j]$ is the number of rooks weakly north-east of the square (i, j). The following criterion for comparing two permutations with respect to the Bruhat order is well known (see e.g. [2, Theorem 2.1.5]).

Lemma 13. *Let $\pi, \rho \in \mathfrak{S}_n$. Then $\pi \leqslant \rho$ if and only if $\pi[i, j] \leqslant \rho[i, j]$ for all $i, j \in [n]$.*

Theorem 4 completely characterises the interesting cases where $\mathfrak{S}(\lambda/\mu)$ is a lower Bruhat interval $[\text{id}, \pi]$. Now we are ready for the proof.

Proof of Theorem 4. We begin with the “only if” direction which is the easier one. For each of the four forbidden patterns we will do the following: First we suppose π contains the pattern. Then we move some of the rooks that constitute the pattern to new positions, and call the resulting rook configuration ρ. This ρ is seen to be covered by $H_R(\pi)$ while $\rho \nleqslant \pi$ in Bruhat order, and we conclude that π is not uniquely maximal in $H_R(\pi)$.

Suppose π contains the pattern 4231 so that there are rooks (i_1, j_4), (i_2, j_2), (i_3, j_3), and (i_4, j_1) with $i_1 < i_2 < i_3 < i_4$ and $j_1 < j_2 < j_3 < j_4$. Move the rooks (i_2, j_2) and (i_3, j_3) to the positions (i_2, j_3) and (i_3, j_2) and call the resulting rook configuration ρ. Then ρ is covered by $H_R(\pi)$ and $\rho > \pi$ in Bruhat order so π is not maximal in $H_R(\pi)$.

Note that the rooks outside the pattern turned out to be irrelevant for the discussion. In fact we could have supposed π was equal to the pattern 4231 and then simply defined $\rho = 4321$. This observation applies to the remaining three patterns as well, and thus the “only if” part of the proof can be written as a table that associates a ρ to each pattern π:

π	ρ
4231	4321
35142	15432
42513	43215
351624	154326

Figure 6 illustrates the table and makes it evident that ρ is covered by $H_R(\pi)$ in each case. That $\rho \nleqslant \pi$ in Bruhat order can be checked easily using Lemma 13.

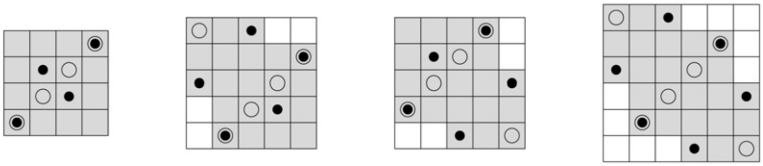


Fig. 6. The dots show the rook configuration π in the four cases 4231, 35142, 42513, and 351624. The shaded squares show the right hull $H_R(\pi)$, and the circles show ρ.

Now it is time to prove the difficult "if" direction. Suppose $\mathfrak{S}(H_R(\pi)) \neq [\mathrm{id}, \pi]$ so that there is a $\rho \in \mathfrak{S}(H_R(\pi))$ with $\rho \nleqslant \pi$. Our goal is to show that π contains some of the four forbidden patterns.

Let the rooks of π and ρ be black and white, respectively. (Observe that some squares may contain both a black and a white rook.) Order the squares $[n]^2$ partially so that $(i, j) \leqslant (i', j')$ if $i \leqslant i'$ and $j \geqslant j'$, i.e. the north-east corner $(1, n)$ is the minimal square of $[n]^2$.

Let L be the set of squares (i, j) with $\rho[i, j] > \pi[i, j]$ and no black rook weakly to the right of (i, j) in row i or above (i, j) in column j. First we show that L is not empty.

Since $\rho \nleqslant \pi$, by Lemma 13 there is a square $(i, j) \in [n]^2$ such that $\rho[i, j] > \pi[i, j]$. Let $(i_{\min}, j_{\min})$ be a minimal square with this property. Then there is no black rook weakly to the right of $(i_{\min}, j_{\min})$ in row $i_{\min}$, for if that were the case the smaller square $(i_{\min} - 1, j_{\min})$ would have the property $\rho[i_{\min} - 1, j_{\min}] > \pi[i_{\min} - 1, j_{\min}]$ as well. Analogously, there is no black rook weakly above $(i_{\min}, j_{\min})$ in column $j_{\min}$. Thus $(i_{\min}, j_{\min})$ belongs to L.

Now we can let $(i_{\max}, j_{\max})$ be a maximal square in L. Since $\rho[i_{\max}, j_{\max}] > \pi[i_{\max}, j_{\max}]$ we have $i_{\max} < n$ and $j_{\max} > 1$. There must be a black rook weakly to the right of $(i_{\max} + 1, j_{\max})$ in row $i_{\max} + 1$ because otherwise the greater square $(i_{\max} + 1, j_{\max})$ would belong to L. By an analogous argument, there is a black rook weakly above $(i_{\max}, j_{\max} - 1)$ in column $j_{\max} - 1$. We have the situation depicted in Fig. 7.

Since there are more white than black rooks inside the rectangle $R = [1, i_{\max}] \times [j_{\max}, n]$ there must also be more white than black rooks inside the diagonally opposite rectangle $R' = [i_{\max} + 1, n] \times [1, j_{\max} - 1]$. In particular there is at least one white rook inside R and at least one white rook inside R'. Since ρ is covered by $H_R(\pi)$ it follows that there is a black rook (i, j) inside R and a black rook (i', j') inside R'; choose (i', j') minimal in R'. Call (i, j) and (i', j') *the witnesses*. Now the situation is exactly as in the proof of Theorem 4.2 in [11] by Gasharov and Reiner. The remaining part of the proof will essentially be a copy of their arguments.

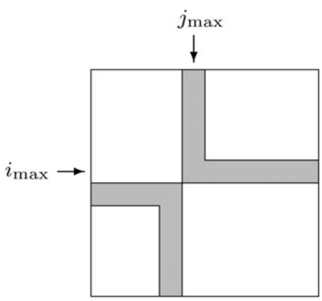


Fig. 7. The shaded region contains no black rook.

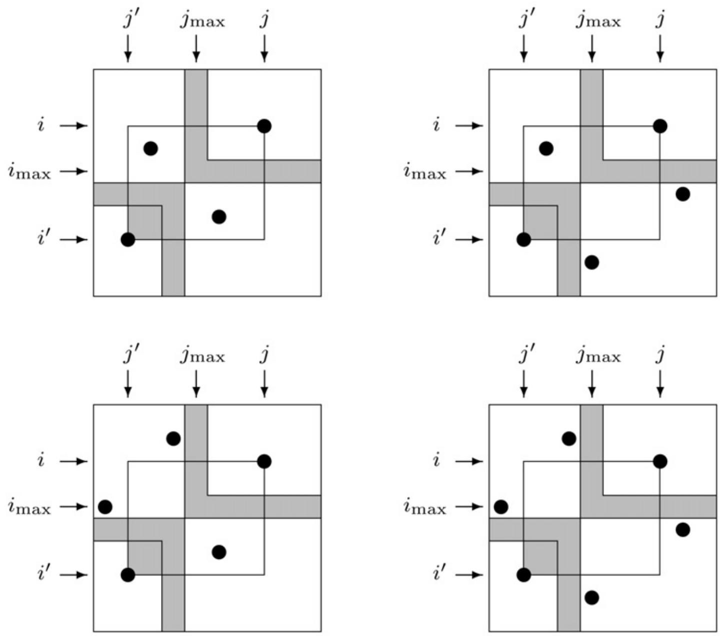


Fig. 8. The four cases of the "if" part of the proof of Theorem 4. The shaded regions contain no black rooks.

We show that at least one of the four forbidden patterns will appear, depending on whether the rectangle $[i, i'] \times [j', j]$ contains a black rook strictly to the left of column $j_{\max}$, and a black rook strictly below row $i_{\max}$. If one can find:

(1) Both, then combining these with the two witnesses produces the pattern 4231 in π. (Look at Fig. 8 for illustrations.)
(2) The former but not the latter, then combining the two witnesses with the former and with the black rooks in column $j_{\max}$ and in row $i_{\max} + 1$ produces the pattern 42513.
(3) The latter but not the former, then combining the two witnesses with the latter and with the black rooks in column $j_{\max} - 1$ and in row $i_{\max}$ produces the pattern 35142.
(4) Neither, then combining the two witnesses with the black rooks in column $j_{\max} - 1$ and $j_{\max}$ and in row $i_{\max}$ and $i_{\max} + 1$ produces the pattern 351624. □

We have proved that a permutation π is the unique (Bruhat) maximal element covered by its right hull, if and only if π avoids four particular patterns. Gasharov and Reiner showed that this pattern avoidance condition holds exactly when the Schubert variety corresponding to $[\mathrm{id}, \pi]$ is *defined by inclusions*. The fact that we were able to reuse a part of Gasharov and Reiner's proof suggests that there might be a shortcut connecting our "maximal in hull" property and Gasharov and Reiner's "defined by inclusions" property without passing through the pattern avoidance condition. On request by an anonymous referee, we have spent quite some time searching for such a shortcut, unfortunately without success. In fact we have come to believe that a connection between Gasharov and Reiner's work and ours necessarily must include some fiddling with diagrams essentially equivalent to our proof above (minus the part that was borrowed from Gasharov and Reiner). Without going into too much detail we want to motivate our belief by comparing Gasharov and Reiner's approach with ours.

Gasharov and Reiner draw permutation diagrams just like we do, though they do not call the dots rooks. They use Fulton's notion of the essential set to characterise permutations defined by inclusions. Put a bubble in every cell in the diagram that does not have rooks weekly to the right or above it. Now the *essential set* contains every bubble that does not have any bubble one step to the left or one step below it. A permutation π is defined by inclusions if and only if, for each bubble (i, j), the rectangle $[1, i] \times [1, j-1]$ contains $\min\{i, j-1\}$ rooks. In our diagrams it is instead the *right hull* of π that is important and, though of course there is some resemblance, we do not see any obvious connection between Gasharov and Reiner's rectangle condition for the essential set and the right hull of the permutation.

5. Poincaré polynomials of A_n

In this section we apply Theorem 4 to the lower Bruhat interval $[\mathrm{id}, w]$ of the symmetric group A_n where w is the maximal minimal coset representative $w = \max A_n^{S\setminus\{s_k\}}$. In the end we obtain the simple formula of Theorem 8.

Let $J^{m,n}$ denote the $m \times n$ matrix with all entries equal to one. The following is a q-analogue of the corollary to Theorem 1 in [5].

Proposition 14. *Let A and B be zero–one matrices of sizes $m \times m$ and $n \times n$, respectively. The block matrix*

$$B \# A \stackrel{\mathrm{def}}{=} \begin{pmatrix} B & J^{n,m} \\ J^{m,n} & A \end{pmatrix}$$

has the $(m+n)$th q-rook number

$$R_{m+n}^{B\#A}(q) = \sum_{i=0}^{\min(m,n)} R_{m-i}^{A}(q) R_{n-i}^{B^{\circlearrowleft}}(q)[i]!_q^2 q^{-i^2}.$$

Proof. It is easy to see that each configuration π of $m+n$ rooks on $B \# A$ is chosen uniquely by the following procedure:

- First, choose a non-negative integer i.
- Then choose a configuration $\mathcal{A}$ of $m-i$ rooks on A and a configuration $\mathcal{B}$ of $n-i$ rooks on B. Together $\mathcal{A}$ and $\mathcal{B}$ form a configuration of $m+n-2i$ rooks on $\left(\begin{smallmatrix} B & 0 \\ 0 & A \end{smallmatrix}\right)$.
- Let X be the $i \times i$ submatrix consisting of the remaining free one-entries of $\left(\begin{smallmatrix} 0 & J^{n,m} \\ 0 & 0 \end{smallmatrix}\right)$, i.e. the one-entries whose row and column have no rook in $\mathcal{A}$ or $\mathcal{B}$. Similarly, let Y be the $i \times i$ submatrix consisting of the remaining free one-entries of $\left(\begin{smallmatrix} 0 & 0 \\ J^{m,n} & 0 \end{smallmatrix}\right)$. Now choose a configuration $\mathcal{X}$ of i rooks on X and a configuration $\mathcal{Y}$ of i rooks on Y.

Let $\mathrm{Inv}(\pi)$ be the set of inversions of π, i.e. pairs (r, r') of rooks such that r is strictly north-east of r'. The number $\mathrm{inv}_A(\mathcal{A})$ counts the cells in A which have no rooks to the right or below. This equals the number of inversions (r, r') such that r belongs to A or $J^{n,m}$ and r' belongs to A or $J^{m,n}$:

$$\mathrm{inv}_A(\mathcal{A}) = \left|\left\{(r, r') \in \mathrm{Inv}(\pi)\colon\ r \in \left(\begin{smallmatrix} 0 & J^{n,m} \\ 0 & A \end{smallmatrix}\right),\ r' \in \left(\begin{smallmatrix} 0 & 0 \\ J^{m,n} & A \end{smallmatrix}\right)\right\}\right|.$$

Similarly, $\mathrm{inv}_{B^{\circlearrowleft}}(\mathcal{B}^{\circlearrowleft})$ counts the cells in B which have no rooks to the left or above, so

$$\mathrm{inv}_{B^{\circlearrowleft}}\left(\mathcal{B}^{\circlearrowleft}\right)=\left|\left\{(r,r')\in\mathrm{Inv}(\pi):\ r\in\left(\begin{smallmatrix}B & J^{n,m}\\ 0 & 0\end{smallmatrix}\right),\ r'\in\left(\begin{smallmatrix}B & 0\\ J^{m,n} & 0\end{smallmatrix}\right)\right\}\right|.$$

We also have

$$\mathrm{inv}_X(\mathcal{X})=\left|\left\{(r,r')\in\mathrm{Inv}(\pi):\ r,r'\in\left(\begin{smallmatrix}0 & 0\\ J^{m,n} & 0\end{smallmatrix}\right)\right\}\right|.$$

and

$$\mathrm{inv}_Y(\mathcal{Y})=\left|\left\{(r,r')\in\mathrm{Inv}(\pi):\ r,r'\in\left(\begin{smallmatrix}0 & J^{n,m}\\ 0 & 0\end{smallmatrix}\right)\right\}\right|.$$

Putting the above equations together yields

$$\begin{aligned}
&\mathrm{inv}_A(\mathcal{A})+\mathrm{inv}_{B^{\circlearrowleft}}\left(\mathcal{B}^{\circlearrowleft}\right)+\mathrm{inv}_X(\mathcal{X})+\mathrm{inv}_Y(\mathcal{Y})\\
&\quad=\mathrm{inv}(\pi)+\left|\left\{(r,r')\in\mathrm{Inv}(\pi):\ r\in\left(\begin{smallmatrix}0 & J^{n,m}\\ 0 & 0\end{smallmatrix}\right),\ r'\in\left(\begin{smallmatrix}0 & 0\\ J^{m,n} & 0\end{smallmatrix}\right)\right\}\right|\\
&\quad=\mathrm{inv}(\pi)+i^2
\end{aligned}\tag{2}$$

where $\mathrm{inv}(\pi)=|\mathrm{Inv}(\pi)|$. Now we exponentiate and sum over all permutations π which can be constructed by the procedure above:

$$\sum_{\pi} q^{\mathrm{inv}(\pi)}=\sum_{i=0}^{\min(m,n)} q^{-i^2}R^{A}_{m-i}(q)R^{B^{\circlearrowleft}}_{n-i}(q)R^{X}_{i}(q)R^{Y}_{i}(q).$$

By Corollary 3, $R^X_i(q)=R^Y_i(q)=[i]!_q$. □

Proof of Theorem 8. A Coxeter system of type $(A_{n-1}, S=\{s_1,s_2,\dots,s_{n-1}\})$ (see Fig. 4) is isomorphic to the symmetric group $\mathfrak{S}_n$ with the adjacent transpositions $s_i=(i\leftrightarrow i+1)$ as generators. A permutation $w\in\mathfrak{S}_n$ can be represented by a rook configuration on $J^{n,n}$ with n rooks, so that $w(i)=j$ precisely if there is a rook in the cell (i,j).

Let w be the maximal element in $A^{S\setminus\{s_k\}}_{n-1}$, i.e. $(w(1),w(2),\dots,w(n))=(n-k+1,n-k+2,\dots,n,1,2,\dots,n-k)$. Using the triangle matrix T_n defined in the end of Section 2, we can write $H_R(w)=(T^{\circlearrowleft}_{n-k}\,\#\,T_k)^{\updownarrow}$ and hence

$$\mathrm{Poin}_{[\mathrm{id},w]}(q)=R_n^{(T^{\circlearrowleft}_{n-k}\#T_k)^{\updownarrow}}(q)=q^{\binom{n}{2}}R_n^{T^{\circlearrowleft}_{n-k}\#T_k}(1/q)$$

which by Proposition 14 equals

$$q^{\binom{n}{2}}\sum_{i=0}^{\min(k,n-k)} R^{T_{n-k}}_{n-k-i}(1/q)R^{T_k}_{k-i}(1/q)[i]!^2_{1/q}q^{i^2}.$$

Using Eq. (1) we obtain

$$P(q)=q^{\binom{n}{2}}\sum_{i=0}^{\min(k,n-k)} q^{-\binom{n-k}{2}}S_{n-k+1,i+1}(1/q)q^{-\binom{k}{2}}S_{k+1,i+1}(1/q)[i]!^2_{1/q}q^{i^2}.$$

Since $[i]!_{1/q}=[i]!_q\cdot q^{-\binom{i}{2}}$ we are done. □

6. The number of elements in some lower intervals of A_n

For a general minimal coset representative $w \in A_{n-1}^{S\setminus\{s_k\}}$, it seems very hard to compute the complete Poincaré polynomial. In this section we will solve the easier problem to determine $\mathrm{Poin}_{[\mathrm{id},w]}(1)$, i.e. the number of elements of $[\mathrm{id}, w]$. We obtain a recurrence relation that allows us to count the elements in polynomial time. In the special case when w is the maximal element in $A_{n-1}^{S\setminus\{s_k\}}$ this method results in a formula different from what we get if we put $q = 1$ in Theorem 8. From this, rather unexpectedly, we obtain a new proof of an identity of Stirling numbers due to Arakawa and Kaneko [1].

The set of minimal coset representatives $A_{n-1}^{S\setminus\{s_k\}}$ consists of the permutations $w \in \mathfrak{S}_n$ with $w(1) < w(2) < \cdots < w(k)$ and $w(k+1) < w(k+2) < \cdots < w(n)$. Such a permutation clearly avoids the patterns in Corollary 5 so the number of elements in the Bruhat interval $[\mathrm{id}, w]$ is given by the nth rook number $R_n^{H_R(w)}$. Fortunately $H_R(w)$ has a simple structure. If $w(k) < n$ then, as can be seen in Fig. 9,

$$\bigl(w\bigl(w(k)+1\bigr), w\bigl(w(k)+2\bigr), \ldots, w(n)\bigr) = \bigl(n - w(k) + 1, n - w(k) + 2, \ldots, n\bigr),$$

so the interval $[\mathrm{id}, w]$ is isomorphic (as a poset) to the interval $[\mathrm{id}, w']$ in $A_{w(k)-1}$, where $w'(i) = w(i)$ for $i = 1, 2, \ldots, w(k)$. Thus we may assume that $w(k) = n$. Then $H_R(w) = \lambda/\mu$, where λ and μ are right-aligned Ferrers matrices with row lengths

$$r_i(\lambda) = \begin{cases} n & \text{if } 1 \leqslant i \leqslant k, \\ n - w(i) + 1 & \text{if } k+1 \leqslant i \leqslant n, \end{cases}$$
$$r_i(\mu) = \begin{cases} n - w(i) & \text{if } 1 \leqslant i \leqslant k, \\ 0 & \text{if } k+1 \leqslant i \leqslant n. \end{cases}$$

For $1 \leqslant i \leqslant k$, let P_i be the $n \times n$ zero–one matrix with ones in the cells $(i,1), (i,2), \ldots, (i, w(i))$. It is easy to see that a rook configuration with n rooks is covered by λ/μ if and only if it is covered by λ and not by any $\lambda - P_i$. Thus, by the principle of inclusion–exclusion we get

$$R_n^{\lambda/\mu} = \sum_{I \subseteq [k]} (-1)^{|I|} R_n^{\lambda - \bigcup_{i\in I} P_i}.$$

By a suitable permutation of the rows, the matrix $\lambda - \bigcup_{i\in I} P_i$ can be transformed to a Ferrers matrix ν with column lengths

$$c_j(\nu) = c_j(\lambda) - \bigl|\bigl\{i \in I\colon\ w(i) \geqslant j\bigr\}\bigr| = c_j(\lambda) - \bigl|w(I) \cap [j, n]\bigr|,$$

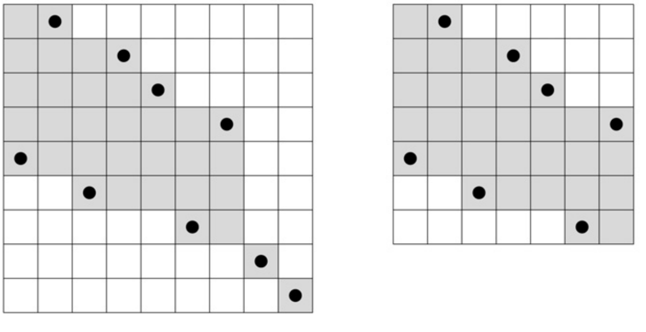


Fig. 9. If $w(k) < n$ as in the left example, we may instead study the smaller example to the right. They have isomorphic lower intervals.

where $w(I) = \{w(i)\colon i \in I\}$ denotes the image of I under w. Theorem 1 with $x = 0$ gives

$$R_n^{\lambda/\mu} = \hat{R}_n^{\lambda/\mu}(0) = \sum_{J \subseteq w([k])} (-1)^{|J|} \prod_{j=1}^{n} \bigl(c_j(\lambda) - \bigl|J \cap [j,n]\bigr| - j + 1\bigr).$$

As we will see in a moment, this expression can be computed efficiently by dynamic programming.

For $1 \leqslant a \leqslant n$ and $0 \leqslant b \leqslant n$, let

$$f(a,b) = \sum_{J \in \binom{w([k]) \cap [a,n]}{b}} \prod_{j=a}^{n} \bigl(c_j(\lambda) - \bigl|J \cap [j,n]\bigr| - j + 1\bigr)$$

where $\binom{w([k]) \cap [a,n]}{b}$ denotes the set of subsets of $w([k]) \cap [a,n]$ of size b. Also put $f(a,b) = 0$ if $b < 0$. It is straightforward to verify the following recurrence relation:

$$\begin{cases} f(a,b) = (c_a(\lambda) - a - b + 1) f(a+1,b) \\ \quad \text{if } n > a \notin w([k]), \\ f(a,b) = (c_a(\lambda) - a - b + 1)(f(a+1,b) + f(a+1,b-1)) \\ \quad \text{if } n > a \in w([k]), \\ f(n,b) = \delta_{b,0}. \end{cases} \tag{3}$$

Here $\delta_{b,0}$ is Dirac's δ-function which is 1 if $b = 0$ and 0 otherwise. Since

$$R_n^{\lambda/\mu} = \sum_{b=0}^{k} (-1)^b f(1,b) \tag{4}$$

the number of elements in $[\mathrm{id}, w]$ is computable in polynomial time.

A special application of the method above admits us to prove our by-product Theorem 10.

Proof of Theorem 10. Consider the case when w is the maximal element in $A_{n-1}^{S \setminus \{s_k\}}$, i.e. $(w(1), w(2), \ldots, w(n)) = (n-k+1, n-k+2, \ldots, n, 1, 2, \ldots, n-k)$. Then $c_a(\lambda) = k + a$ if $a \leqslant n-k$ and $c_a(\lambda) = n$ if $a \geqslant n-k+1$, so the recurrence (3) becomes

$$\begin{cases} f(a,b) = (k-b+1) f(a+1,b) & \text{if } a \leqslant n-k, \\ f(a,b) = (n-a-b+1)(f(a+1,b) + f(a+1,b-1)) & \text{if } n-k+1 \leqslant a \leqslant n-1, \\ f(n,b) = \delta_{b,0}. \end{cases}$$

Iteration of the first line of the recurrence yields

$$f(1,b) = (k-b+1)^{n-k} f(n-k+1,b). \tag{5}$$

Putting $g(\alpha,\beta) = f(n-\alpha+1, \alpha-\beta)$ for $1 \leqslant \alpha \leqslant k$, our recurrence transforms to

$$\begin{cases} g(\alpha,\beta) = \beta \cdot (g(\alpha-1,\beta) + g(\alpha-1,\beta-1)) & \text{if } 2 \leqslant \alpha \leqslant k, \\ g(1,\beta) = \delta_{\beta,1}. \end{cases}$$

We recognise this as the recurrence for $\beta! S_{\alpha,\beta}$ where $S_{\alpha,\beta}$ are Stirling numbers of the second kind; thus $f(a,b) = (n-a-b+1)! S_{n-a+1,n-a-b+1}$ for $n-k+1 \leqslant a \leqslant n$. Combining this with Eq. (5) and plugging the result into Eq. (4), we obtain

$$R_n^{\lambda/\mu} = \sum_{b=0}^{k} (-1)^b (k-b+1)^{n-k} (k-b)! S_{k,k-b}$$

which also can be written as

$$(-1)^k \sum_{i=0}^{k} (-1)^i (i+1)^{n-k} i! S_{k,i}.$$

This happens to be the formula for the poly-Bernoulli number B_n^{k-n} defined by Kaneko [15]. □

7. Type B

In this section we compute the Poincaré polynomial of the lower Bruhat interval $[\mathrm{id}, w]$ in the hyperoctahedral group B_n, where w is the maximal minimal coset representative, $w = \max B_n^{S \setminus \{s_0\}}$.

We will represent B_n combinatorially by the set $\mathfrak{S}_n^B \overset{\text{def}}{=} \{\pi \in \mathfrak{S}_{2n}\colon \pi^{\circlearrowleft} = \pi\}$ of rotationally symmetric maximal rook configurations on $J^{2n,2n}$, see [2, Chapter 8]. In this representation $\pi \leqslant \rho$ in Bruhat order on B_n if and only if $\pi \leqslant \rho$ as elements of $\mathfrak{S}_{2n}$ [2, Corollary 8.1.9]. The rank of π is

$$\ell(\pi) = \bigl(\mathrm{inv}(\pi) + \mathrm{neg}(\pi)\bigr)/2 \tag{6}$$

where $\mathrm{inv}(\pi)$ is the usual inversion number of π as an element of $\mathfrak{S}_{2n}$, and $\mathrm{neg}(\pi) \overset{\text{def}}{=} |\{i \in [n+1, 2n]\colon \pi(i) \leqslant n\}|$, see [2, Chapter 8, Exercise 2].

For a zero–one matrix A of size $2n \times 2n$, let

$$\mathrm{RB}^A(q,t) \overset{\text{def}}{=} \sum_{\pi \in \mathfrak{S}_n^B \cap \mathfrak{S}(A)} q^{\mathrm{inv}(\pi)} t^{\mathrm{neg}(\pi)}.$$

Proposition 15. *Let A be a zero–one matrix of size $n \times n$. Then*

$$\mathrm{RB}^{A^{\circlearrowleft} \# A}(q,t) = \sum_{i=0}^{n} R_{n-i}^A\bigl(q^2\bigr)[i]!_{q^2} q^{-i^2} t^i.$$

Proof. The proof is almost identical to the proof of Proposition 14.

When $m = n$, it is easy to see that the permutation π constructed by the procedure in the proof of Proposition 14 is rotationally symmetric if and only if $\mathcal{B} = \mathcal{A}^{\circlearrowleft}$ and $\mathcal{Y} = \mathcal{X}^{\circlearrowleft}$. Thus, putting $m := n$, $B := A^{\circlearrowleft}$, $Y := X^{\circlearrowleft} = X$, $\mathcal{B} := \mathcal{A}^{\circlearrowleft}$, and $\mathcal{Y} := \mathcal{X}^{\circlearrowleft}$ into Eq. (2) and using the identity $(A^{\circlearrowleft})^{\circlearrowleft} = A$, we obtain

$$2\bigl(\mathrm{inv}_A(\mathcal{A}) + \mathrm{inv}_X(\mathcal{X})\bigr) = \mathrm{inv}(\pi) + i^2.$$

Obviously, $\mathrm{neg}(\pi) = i$. Exponentiation and summation over all rotationally symmetric permutations π on $A^{\circlearrowleft} \# A$ yields

$$\sum_{\pi} q^{\mathrm{inv}(\pi)} t^{\mathrm{neg}(\pi)} = \sum_{i=0}^{n} q^{-i^2} R_{n-i}^A\bigl(q^2\bigr) R_i^X\bigl(q^2\bigr) t^i.$$

By Corollary 3, $R_i^X(q^2) = [i]!_{q^2}$. □

Now we are ready for the proof of Theorem 9.

Proof of Theorem 9. From Theorem 4 and Eq. (6) we obtain

$$\mathrm{Poin}_{[\mathrm{id},w]}(q) = \sum_{u\in[\mathrm{id},w]} q^{\ell(u)} = \mathrm{RB}^{H_R(w)}\big(q^{1/2}, q^{1/2}\big).$$

It is easy to see that $H_R(w) = (T_n^{\circlearrowleft} \# T_n)^{\updownarrow}$. Hence

$$\mathrm{Poin}_{[\mathrm{id},w]}(q) = \mathrm{RB}^{(T_n^{\circlearrowleft}\#T_n)^{\updownarrow}}\big(q^{1/2}, q^{1/2}\big) = q^{n^2}\,\mathrm{RB}^{(T_n^{\circlearrowleft}\#T_n)}\big(q^{-1/2}, q^{-1/2}\big)$$

where we have used the fact that

$$\mathrm{RB}^{A^{\updownarrow}}(q,t) = q^{\binom{2n}{2}} t^n\,\mathrm{RB}^A\big(q^{-1}, t^{-1}\big)$$

for any $2n \times 2n$ zero–one matrix A. By Proposition 15 we can now compute

$$\mathrm{Poin}_{[\mathrm{id},w]}(q) = q^{n^2}\sum_{i=0}^{n} R_{n-i}^{T_n}(1/q)[i]!_{1/q}\, q^{\binom{i}{2}}.$$

Using Eq. (1) and the identity $[i]!_{1/q} q^{\binom{i}{2}} = [i]!_q$, we finally obtain

$$\mathrm{Poin}_{[\mathrm{id},w]}(q) = q^{\binom{n+1}{2}}\sum_{i=0}^{n} S_{n+1,i+1}(1/q)[i]!_q. \qquad \square$$

8. Open problems

Perhaps the reason we still do not know much about the Poincaré polynomials of Bruhat intervals after several decades of research in the area is the lack of natural methods to attack it. We hope the framework and the tools presented here will make the problem more accessible, and we would like to suggest a number of interesting open questions.

- What is the Poincaré polynomial $\mathrm{Poin}_{[\mathrm{id},w]}(q)$ in the even-signed permutation group D_n if $w = \max D_n^{S\setminus\{s_0\}}$ is the maximal minimal representative in the quotient modulo a maximal parabolic subgroup isomorphic to A_{n-1}?
- What is $\mathrm{Poin}_{[\mathrm{id},w]}(q)$ in the affine group $\tilde{A}_n$?
- Are there formulas for the generalised Eulerian polynomial $\sum_{v\in[\mathrm{id},w]} t^{d(v)}$ or even for the bivariate generating function $\sum_{v\in[\mathrm{id},w]} t^{d(v)} q^{\ell(v)}$, where $d(v) = |\{s \in S\colon \ell(vs) < \ell(v)\}|$ is the *descent number* of v?
- In a recent paper by Björner and Ekedahl [3] it is shown (for any crystallographic Coxeter group) that $0 \leqslant i < j \leqslant \ell(w) - i$ implies $f_i^w \leqslant f_j^w$, where f_i^w is the q^i-coefficient of $\mathrm{Poin}_{[\mathrm{id},w]}(q)$. Perhaps one can say more about the particular Poincaré polynomials discussed in the present paper. Computer experiments support the following conjecture:

  Conjecture 16. *The Poincaré polynomials of Theorems* 8 *and* 9 *are unimodal.*

- As noted by Gasharov and Reiner [11, p. 559], Ding's partition varieties [8] correspond not only to certain Bruhat intervals of the whole group A_n but also to some intervals of the quotient $A_n/(A_n)_J$ for certain parabolic subgroups $(A_n)_J$. Can something similar be done in the more general setting of skew partition varieties?

- Given a polynomial, what board (i.e. zero–one matrix), if any, has it as its rook polynomial? In 1970 Foata and Schützenberger [9] showed that distinct increasing Ferrers boards have distinct rook polynomials, and in a recent paper [18] Mitchell characterised these polynomials in terms of roots and divisibility. What is true for skew Ferrers boards?
- Develin [7] classified the isomorphism types of lower Bruhat intervals of 312-avoiding permutations by using the connection to rook posets discovered by Ding [8]. What are the isomorphism types of lower Bruhat intervals of permutations that avoid the patterns 4231, 35142, 42513, and 351624?

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